UNIVERSAL BOUNDS FOR TRACES OF THE DIRICHLET LAPLACE OPERATOR

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ABSTRACT. We derive upper bounds for the trace of the heat kernel $Z(t)$ of the Dirichlet Laplace operator in an open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$. In domains of finite volume the result improves an inequality of Kac. Using the same methods we give bounds on $Z(t)$ in domains of infinite volume.

For domains of finite volume the bound on $Z(t)$ decays exponentially as t tends to infinity and it contains the sharp first term and a correction term reflecting the properties of the short time asymptotics of $Z(t)$. To prove the result we employ refined Berezin-Li-Yau inequalities for eigenvalue means.

1. INTRODUCTION AND MAIN RESULTS

Let Ω be an open subset of $\mathbb{R}^d$, $d \geq 2$. Consider the Laplace operator $-\Delta_\Omega$ on Ω subject to Dirichlet boundary conditions defined in the form sense on the form domain $H_0^1(\Omega)$. If the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, e.g. if the volume of Ω is finite, the spectrum of $-\Delta_\Omega$ is discrete and consists of a monotone sequence of positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ accumulating at infinity. We count these eigenvalues according to their multiplicity.

The main goal of this paper is to derive some new universal upper bounds for the trace of the heat kernel

$$Z(t) \,=\, \operatorname{Tr}\left(e^{+\Delta_\Omega t}\right) \,=\, \sum_k e^{-\lambda_k t}$$

which are valid for arbitrary open sets $\Omega \subset \mathbb{R}^d$ with finite volume $|\Omega|$ and for all $t > 0$. The first and most fundamental bound of this type is due to M. Kac, [Kac51]. He proved that for any open domain $\Omega \subset \mathbb{R}^d$ and all $t > 0$ the estimate

$$Z(t) \,\leq\, \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \tag{1}$$

holds true. This bound is sharp in the sense that it reflects the leading term of the short time asymptotics of the function $Z(t)$, see [Min54, Kac66]

$$Z(t) \,=\, \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \quad \text{as} \quad t \to 0+\,. \tag{2}$$

Several improvements of (1) are known, e.g. see [vdB84b, FLV95, Dav85, Dav89, Sim83, vdB84a] and further references therein. For example, M. van den Berg proved in [vdB87], that if Ω is a connected region with a smooth boundary $\partial\Omega$ and a surface area $|\partial\Omega|$, then

$$\left| Z(t) - \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} + \frac{|\partial\Omega|}{4\,(4\pi t)^{\frac{d-1}{2}}} \right| \,\leq\, \frac{d^4}{\pi^{\frac{d}{2}}} \frac{|\Omega|}{t^{\frac{d}{2}-1} R^2}, \quad t > 0,$$

where the constant R depends on properties of $\partial\Omega$. This estimate contains even the second term of the short time asymptotic expansion of $Z(t)$, see [MS67, Smi81, BC86] and [Bro93]. Most of these results are based on a probabilistic approach and implement local estimates for the heat kernel. Therefore one has to impose appropriate conditions on Ω and on its boundary $\partial\Omega$.

We use a different approach based on some refined spectral estimates for the Riesz means

$$R_\sigma(\Lambda) \,=\, \mathrm{Tr}\,(-\Delta_\Omega - \Lambda)_-^\sigma \,=\, \sum_k (\Lambda - \lambda_k)_+^\sigma\,, \quad \Lambda > 0.$$

For these objects the fundamental bounds are given by the Berezin-Li-Yau inequalities

$$R_\sigma(\Lambda) \,\le\, L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+\frac{d}{2}}, \quad \sigma \ge 1\,,\ \Lambda > 0, \tag{3}$$

where

$$L^{cl}_{\sigma,d} \,=\, \frac{\Gamma(\sigma+1)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\sigma+\frac{d}{2}+1\right)}.$$

This result is sharp as well in the sense that the bound captures the first term of the high energy asymptotics

$$R_\sigma(\Lambda) \,=\, L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+\frac{d}{2}} + o\left(\Lambda^{\sigma+\frac{d}{2}}\right) \quad \text{as} \quad \Lambda \to +\infty.$$

Via Laplace transformation - and reversely via Tauberian theorems - this asymptotic formula is closely connected with (2). On the level of uniform inequalities one can deduce Kac' inequality on $Z(t)$ from Berezin-Li-Yau bounds. Reversely, to recover sharp Berezin-Li-Yau bounds from Kac' inequality one needs some additional information. For example, in [HH07] Harrell and Hermi formally deduced Berezin-Li-Yau bounds for $\sigma \ge 2$ from Kac' inequality based on a monotonicity result by Harrell and Stubbe.[1] Similar arguments fail for $\sigma < 2$.

While both (3) and (1) are sharp in the sense that they capture the main asymptotic behaviour and therefore constants in these inequalities cannot be improved, one can expect that more subtle bounds might invoke additional lower order correction terms. Indeed, we know that under certain conditions on the geometry of Ω the asymptotics

$$R_\sigma(\Lambda) \,=\, L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+\frac{d}{2}} - \frac{1}{4}\,L^{cl}_{\sigma,d-1}\,|\partial\Omega|\,\Lambda^{\sigma+\frac{d-1}{2}} + o\left(\Lambda^{\sigma+\frac{d-1}{2}}\right)$$

holds true as $\Lambda \to \infty$, see [Ivr98]. Recently there have been several results on semciclassical inequalities improving (3) with negative correction terms of lower order, reflecting the effect of the second term of the asymptotics, see [Mel03, Wei08, KVW08], and [FLU02] for discrete operators.

Let us first point out a result of Melas. In [Mel03] he effectively showed that[2]

$$R_1(\Lambda) \,\le\, L^{cl}_{1,d}\,|\Omega|\left(\Lambda - M_d\frac{|\Omega|}{I(\Omega)}\right)_+^{1+\frac{d}{2}} \tag{4}$$

[1] One should mention, that in fact, due to Weyl's asymptotic law, the monotonicity result implies sharp Berezin-Li-Yau bounds for $\sigma \ge 2$ on its own.

[2] This inequality is in fact the Legendre transform of Melas' result.

holds for $\Lambda > 0$. Again applying Laplace transformation Harrell and Hermi deduced an improvement of Kac' inequality [HH07]

$$Z(t) \leq \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \exp\left(-M_d \frac{|\Omega|}{I(\Omega)}\, t\right), \tag{5}$$

where $I(\Omega) = \min_{a\in\mathbb{R}^d} \int_\Omega |x-a|^2\, dx$ and M_d is a constant depending only on the dimension. This improvement holds true for all $t > 0$ and any open set Ω with finite volume - without any conditions on the boundary $\partial\Omega$. These authors conjecture also that (5) can be improved to

$$Z(t) \leq \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{t}{|\Omega|^{\frac{2}{d}}}\right) \tag{6}$$

for all $t > 0$ and all open sets Ω of finite volume. Asymptotic considerations show that this conjecture is plausible for small t as well as for large t. However, one should mention, that neither the correction term in (4) is of the expected order for high energies, nor is the improvement (5) or even the conjecture (6) of correct order for small $t > 0$.

To derive universal bounds on $Z(t)$ like (6) depending only on the volume of Ω and not including any further geometrical information one can employ an isoperimetric result due to Luttinger [Lut73]. He shows that Steiner-symmetrization of an open set Ω increases the trace of the heat kernel in this set. Thus for any open set $\Omega \subset \mathbb{R}^d$ with finite volume the inequality

$$Z(t) \leq Z^*(t) \tag{7}$$

holds true for all $t > 0$, where $Z^*(t)$ denotes the trace of the heat kernel in the ball $B \subset \mathbb{R}^d$ with the same volume as Ω.

Here we prove a refined universal bound on $Z(t)$ reflecting the correct asymptotic properties. To this end we shall follow the approach in [Wei08]. There a Berezin-Li-Yau type bound on R_σ for $\sigma \geq 3/2$ with a correction term of the expected order has been found, see inequality (18) below. Using the same method we prove a refined Berezin-Li-Yau inequality, see Proposition 5, that gives rise to an improved bound on $Z(t)$ applicable to any open set Ω with finite volume. This bound decays exponentially as t tends to infinity and contains a negative correction term of correct order as t tends to zero.

Moreover, we can consider unbounded domains $\Omega \subset \mathbb{R}^d$ with infinite volume. While the results of Kac and Luttinger must fail for such domains, we show that under appropriate conditions on Ω our refined inequalities can still be applied and give order-sharp upper bounds.

This paper is structured as follows: In section 2 we state the main results. Then in section 3 we provide some auxiliary notation and auxiliary results including improved Berezin-Li-Yau inequalities. In section 4 we prove Theorem 1 and compare this result to other bounds on $Z(t)$. In section 5 we discuss some applications to unbounded domains and domains with infinite volume. Finally, in section 6 we apply a method by M. Aizenmann and E. H. Lieb [AL78] to the results from section 3 in order to prove refined bounds on the eigenvalue means $R_\sigma(\Lambda)$.

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2. Main Results

To state the main result we have to introduce some auxiliary notation. Let $\Gamma(z)$ be the usual Gamma-function and by

$$\tilde{\Gamma}(z, s_1, s_2) = \int_{s_1}^{s_2} s^{z-1} e^{-s} ds / \Gamma(z)$$

we denote normed incomplete Gamma-functions. If $s_1 = 0$ we write $\tilde{\Gamma}(z, s) = \tilde{\Gamma}(z, 0, s)$ and $\hat{\Gamma}(z, s) = 1 - \tilde{\Gamma}(z, s) = \tilde{\Gamma}(z, s, +\infty)$. Note that for $a > 0$ we have

$$\tilde{\Gamma}(a, t) = \frac{t^a}{a\,\Gamma(a)} + O\left(t^{a+1}\right) \quad \text{as} \quad t \to 0+ \ \text{ and} \tag{8}$$

$$\hat{\Gamma}(a, t) = \frac{t^{a-1}}{\Gamma(a)} \exp(-t) + O\left(t^{a-2} \exp(-t)\right) \quad \text{as} \quad t \to \infty. \tag{9}$$

Furthermore, let $B(\alpha, \beta)$ be the usual Beta-function. By

$$\tilde{B}(s_1, s_2, \alpha, \beta) = \int_{s_1}^{s_2} s^{\alpha-1} (1-s)^{\beta-1} ds / B(\alpha, \beta)$$

we denote normed incomplete Beta-functions and for $s_1 = 0$ we write in short $\tilde{B}(s, \alpha, \beta) = \tilde{B}(0, s, \alpha, \beta)$ and $\hat{B}(s, \alpha, \beta) = 1 - \tilde{B}(s, \alpha, \beta) = \tilde{B}(s, 1, \alpha, \beta)$. Note that for $\alpha, \beta > 0$ we have

$$B(0, t, \alpha, \beta) = \frac{1}{\alpha} t^{\alpha} + O\left(t^{\alpha+1}\right) \quad \text{as} \quad t \to 0+\ . \tag{10}$$

Next we remark that in view of the isoperimetric inequality by Rayleigh, Faber and Krahn [Fab23, Kra25] on the ground state λ_1 we can always choose

$$\tilde{\lambda} = \frac{\pi}{\Gamma\left(\frac{d}{2}+1\right)^{2/d}} \frac{j^2_{\frac{d}{2}-1,1}}{|\Omega|^{2/d}} \ \leq\ \lambda_1 \tag{11}$$

as a lower bound on λ_1, where $j_{k,1}$ denotes the first zero of the Bessel-function J_k.

For $r \in \mathbb{R}$ put $(r)_+ = \max\{r, 0\}$ and for $d \in \mathbb{N}$ let

$$\sigma_d \ = \begin{cases} 5/2 & \text{if} \quad d = 2 \\ 2 & \text{if} \quad d = 3 \\ 3/2 & \text{if} \quad d \geq 4 \end{cases} . \tag{12}$$

Finally, let $\Omega \subset \mathbb{R}^d$ be an arbitrary open set with finite volume $|\Omega|$.

Theorem 1. *Let $\lambda \in [\tilde{\lambda}, \lambda_1]$. For any $t > 0$ the bound*

$$Z(t) \ \leq \ \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \hat{\Gamma}\left(\sigma_d + \frac{d}{2} + 1, \lambda t\right) \ - \ (R(t, \lambda))_+$$

holds true with a remainder term

$$R(t) = c_{1,d} \frac{|\Omega|^{\frac{d-1}{d}}}{(4\pi t)^{\frac{d-1}{2}}} \hat{\Gamma}\left(\sigma_d + \frac{d+1}{2}, \lambda t\right) - c_{2,d} \frac{|\Omega|^{\frac{d-3}{d}}}{(4\pi t)^{\frac{d-3}{2}}} \hat{\Gamma}\left(\sigma_d + \frac{d-1}{2}, \lambda t\right) ,$$

where

$$c_{1,d} = \frac{B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right)}{2} \frac{\Gamma\left(\frac{d}{2}+1\right)^{\frac{d-1}{d}}}{\Gamma\left(\frac{d+1}{2}\right)} \quad \textit{and}$$

$$c_{2,d} = \frac{\pi^2(d-1)B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right)}{96(2\sigma_d + d - 1)} \frac{\Gamma\left(\frac{d}{2}+1\right)^{\frac{d-3}{d}}}{\Gamma\left(\frac{d+1}{2}\right)} .$$

Remark. Because of (8) Theorem 1 can then be read as

$$Z(t) \le \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} - c_{1,d} \frac{|\Omega|^{\frac{d-1}{d}}}{(4\pi t)^{\frac{d-1}{2}}} - r(t) \tag{13}$$

with an explicit remainder term $r(t) = O(t^{-\frac{d-3}{2}})$ as $t \to 0+$. We note that the bound captures the main asymptotic behaviour of $Z(t)$ as t tends to zero: The first term equals the leading term of the short time asymptotics of $Z(t)$ and the second term shows the correct order in t compared with the second term of the asymptotic expansion.

Moreover, note that in view of (9) the bound from Theorem 1 decays exponentially as t tends to infinity. More precisely, it follows that the bound is of order $O(t^{\sigma_d+1}\exp(-\tilde{\lambda}\, t))$ as $t \to \infty$.

Remark. If we choose $\lambda = \tilde{\lambda}$ introduced in (11) we arrive at a universal upper bound on $Z(t)$ depending only on $|\Omega|$ and not including any explicit information on λ_1. For the explicit statement see Corollary 9 in section 4. This result implies the conjectured inequality (6) for dimensions $d \le 633$.

As stated above, our proof of Theorem 1 relies on improved bounds for Riesz means of eigenvalues. Let us state the corresponding result.

Theorem 2. *Let $\lambda \in [\tilde{\lambda}, \lambda_1]$ and $\sigma > \sigma_d$ and put $\tau_\Omega = \frac{\pi^2 d^2}{|\Omega|^{\frac{2}{d}}}$. Then the estimate*

$$R_\sigma(\Lambda) \le L^{cl}_{\sigma,d}\, |\Omega|\, \hat{B}\left(\frac{\lambda}{\Lambda}, \sigma_d + \frac{d}{2} + 1, \sigma - \sigma_d\right) \Lambda^{\sigma+\frac{d}{2}} - (S(\Lambda,\lambda))_+$$

holds true for all $\Lambda \ge \lambda$, where

$$S(\Lambda,\lambda) = L^{cl}_{\sigma,d-1}\, |\Omega|^{\frac{d-1}{d}}\, \Lambda^{\sigma+\frac{d-1}{2}} \frac{B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right)}{2} \hat{B}\left(\frac{\lambda}{\Lambda}, \sigma_d + \frac{d+1}{2}, \sigma - \sigma_d\right) \tag{14}$$

if $\lambda \ge \tau_\Omega$,

$$S(\Lambda,\lambda) = L^{cl}_{\sigma,d}\, |\Omega|\, \Lambda^{\sigma+\frac{d}{2}} \frac{1}{d} \hat{B}\left(\frac{\lambda}{\Lambda}, \sigma_d + \frac{d}{2} + 1, \sigma - \sigma_d\right) \tag{15}$$

if $\lambda < \tau_\Omega$ and $\Lambda < \tau_\Omega$, or

$$\begin{aligned} S(\Lambda,\lambda) = {} & L^{cl}_{\sigma,d-1}\, |\Omega|^{\frac{d-1}{d}} \Lambda^{\sigma+\frac{d-1}{2}} \frac{B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right)}{2} \hat{B}\left(\frac{\tau_\Omega}{\Lambda}, \sigma_d + \frac{d+1}{2}, \sigma - \sigma_d\right) \\ & + L^{cl}_{\sigma,d}\, |\Omega|\, \Lambda^{\sigma+\frac{d}{2}} \frac{1}{d} \tilde{B}\left(\frac{\lambda}{\Lambda}, \frac{\tau_\Omega}{\Lambda}, \sigma_d + \frac{d}{2} + 1, \sigma - \sigma_d\right) , \end{aligned} \tag{16}$$

if $\lambda < \tau_\Omega$ and $\Lambda \ge \tau_\Omega$.

Remark. Again we can choose λ as in (11) and we arrive at a universal bound depending only on $|\Omega|$.

Remark. In view of (10) Theorem 2 can be read as

$$R_\sigma(\Lambda) \;\le\; L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+\frac{d}{2}} - \frac{1}{2}\,B\left(\frac{1}{2},\sigma_d+\frac{d+1}{2}\right)\,L^{cl}_{\sigma,d}\,|\Omega|^{\frac{d-1}{d}}\,\Lambda^{\sigma+\frac{d-1}{2}} + s(\Lambda)$$

with an explicit remainder term $s(\Lambda) = O\left(\Lambda^{-1}\right)$ as $\Lambda \to \infty$.

3. Notation and auxiliary results

Fix a Cartesian coordinate system in $\mathbb{R}^d$ and write $x = (x', x_d) \in \mathbb{R}^{d-1}\times\mathbb{R}$ for $x \in \mathbb{R}^d$. For a given $\Lambda > 0$ define

$$l_\Lambda \;=\; \pi\Lambda^{-\frac{1}{2}}.$$

Now consider an open set $\Omega \subset \mathbb{R}^d$. Each section $\Omega(x') = \{x_d \in \mathbb{R} : (x', x_d) \in \Omega\}$ is a one-dimensional open set and consists of at most countably many open disjoint intervals $J_k(x')$, $k = 1, \ldots, N(x') \le \infty$. Let $\kappa(x', \Lambda) \subset \mathbb{N}$ be the subset of all those indices k, for which the corresponding interval $J_k(x')$ is strictly longer than l_Λ. The number these indices is denoted by $\chi(x', \Lambda)$. Put

$$\Omega_\Lambda(x') = \bigcup_{k\in\kappa(x',\Lambda)} J_k(x') \quad \text{and} \quad \Omega_\Lambda = \bigcup_{x'\in\mathbb{R}^{d-1}} \{x'\}\times\Omega_\Lambda(x')\,.$$

Obviously the set Ω_Λ is the subset of Ω, where Ω is "wide enough" in x_d-direction. The quantity

$$d_\Lambda(\Omega) = \int_{\mathbb{R}^{d-1}} \chi\left(x', \Lambda\right) dx'$$

is an effective area of the projection of Ω_Λ onto the $d-1$-dimensional hyperplane $(x', 0)$ counting also the multiplicities of the sufficiently long intervals $J_k(x')$.

Moreover, for $\mu \ge 2$ put

$$\varepsilon(\mu) = \inf_{A\ge 1}\left(\int_0^A \left(1-\tfrac{t^2}{A^2}\right)_+^\mu dt - \sum_{k\ge 1}\left(1-\tfrac{k^2}{A^2}\right)_+^\mu\right) > 0\,. \tag{17}$$

We are now in the position to state the improved Berezin-Li-Yau bound from [Wei08]:

Proposition 3. *For any open domain $\Omega \subset \mathbb{R}^d$, $\sigma \ge 3/2$ and all $\Lambda > 0$ the bound*

$$R_\sigma(\Lambda) \;\le\; L^{cl}_{\sigma,d}\,|\Omega_\Lambda|\,\Lambda^{\sigma+\frac{d}{2}} \;-\; \varepsilon\left(\sigma+\tfrac{d-1}{2}\right)\,L^{cl}_{\sigma,d-1} d_\Lambda(\Omega)\,\Lambda^{\sigma+\frac{d-1}{2}} \tag{18}$$

holds true.

Let us state also the following result on the explicit values of $\varepsilon(\mu)$.

Lemma 4. *For all $\mu \ge 3$ we have*

$$\varepsilon\left(\mu\right) \;=\; \frac{1}{2}\,B\left(\frac{1}{2}, \mu+1\right).$$

Proof. In view of definition (17) and the identity

$$\int_0^A \left(1-\frac{t^2}{A^2}\right)_+^\mu dt \;=\; \frac{A}{2}B\left(\frac{1}{2},\mu+1\right)$$

we have to show that

$$\sum_{k\ge 1}\left(1-\frac{k^2}{A^2}\right)_+^\mu \le \frac{A-1}{2}\,B\left(\frac{1}{2},\mu+1\right)$$

holds true for $\mu \geq 3$ and $A \geq 1$.

For $\mu = 3$ the claim can be checked by elementary analytic methods, since there is an explicit expression for the sum in terms of A and its integer part.

To deduce the estimate for $\mu > 3$, we start with the identity [AL78]

$$\sum_{k\geq 1}\left(1-\frac{k^2}{A^2}\right)_+^{\mu} = \frac{1}{A^{2\mu}}\frac{1}{B(4,\mu-3)}\int_0^{A^2-1}\tau^{\mu-4}\left(A^2-\tau\right)^3\sum_{k\geq 1}\left(1-\frac{k^2}{A^2-\tau}\right)_+^3 d\tau$$

and estimate

$$\begin{aligned}\sum_{k\geq 1}\left(1-\frac{k^2}{A^2}\right)_+^{\mu} &\leq \frac{1}{A^{2\mu}}\frac{B\left(\frac12,4\right)}{B(4,\mu-3)}\int_0^{A^2-1}\tau^{\mu-4}\left(A^2-\tau\right)^3\frac{\left(A^2-\tau\right)^{\frac12}-1}{2}\,d\tau\\ &= \frac{1}{2A^{2\mu}}\frac{B\left(\frac12,4\right)}{B(4,\mu-3)}\int_0^{A^2-1}\tau^{\mu-4}\left(\left(A^2-\tau\right)^{\frac72}-\left(A^2-\tau\right)^3\right)d\tau\,.\end{aligned}$$

If we substitute $s = \frac{\tau}{A^2}$, we see that the last integral equals

$$A^{2\mu}\int_0^{1-A^{-2}} s^{\mu-4}\left(A(1-s)^{\frac72}-(1-s)^3\right)ds =$$

$$A^{2\mu}\left(AB\left(\mu-3,\frac92\right)-B\left(4,\mu-3\right)-\int_{1-A^{-2}}^1 s^{\mu-4}\left(A(1-s)^{\frac72}-(1-s)^3\right)ds\right).$$

Now we can use the identity $B\left(\mu-3,\frac92\right)B\left(\frac12,4\right)/B\left(4,\mu-3\right) = B\left(\frac12,\mu+1\right)$ and substitute $t = 1-s$ to conclude

$$\begin{aligned}\sum_{k\geq 1}\left(1-\frac{k^2}{A^2}\right)_+^{\mu} &\leq \frac{A}{2}B\left(\frac12,\mu+1\right)-\frac{B\left(\frac12,4\right)}{2B(4,\mu-3)}\\ &\quad\times\left(B\left(4,\mu-3\right)-\int_0^{A^{-2}}(1-t)^{\mu-4}\,t^3\left(1-A\sqrt{t}\right)dt\right).\end{aligned}$$

It remains to remark that the inequality

$$\frac{B\left(\frac12,4\right)}{B(4,\mu-3)}\left(B\left(4,\mu-3\right)-\int_0^{A^{-2}}(1-t)^{\mu-4}\,t^3\left(1-A\sqrt{t}\right)dt\right) \geq B\left(\frac12,\mu+1\right),$$

holds true for all $A \geq 1$, since we have equality in the case $A = 1$ and since the left hand side is non-decreasing in $A \geq 1$. $\square$

In fact, we shall need a modified version of Proposition 3.

Let $p_d(x';\Omega) = |\Omega(x')|_1$ be the one-dimensional Lebesgue measure of $\Omega(x')$, that is the aggregated length of all intervals forming $\Omega(x')$. Since Ω is open, the function $p_d(x';\Omega)$ is Lebesgue measurable, and we can define the distribution function [3]

$$m_d(\tau;\Omega) = \left|\{x' : p_d(x';\Omega) > \tau\}\right|_{d-1}, \quad \tau > 0.$$

It is non-negative, non-increasing, continuous from the right and it satisfies the identity

$$\int_0^\infty m_d(\tau;\Omega)\,d\tau = |\Omega|. \tag{19}$$

[3] Here $|\cdot|_{d-1}$ stands for the Lebesgue measure in the dimension $d-1$.

We interchange now the roles of x_d and x_i for $i = 1, \dots, d-1$ and introduce in the same way the distribution functions $m_i(\cdot;\Omega)$ for Ω measured along the x_i-axes. Finally, put

$$M_i(y;\Omega) \;=\; \int_0^y m_i(\tau;\Omega)\,d\tau \quad \text{for} \quad i = 1, \dots, d.$$

With this notation we can formulate a result similar to (18):

Proposition 5. *For any open domain $\Omega \subset \mathbb{R}^d$, $\sigma \geq 3/2$ and all $\Lambda > 0$*

$$R_\sigma(\Lambda) \;\leq\; L^{cl}_{\sigma,d} \int_{\frac{\pi}{\sqrt{\Lambda}}}^{\infty} m_i(\tau;\Omega)\,d\tau\,\Lambda^{\sigma+\frac{d}{2}} + \delta_{\sigma,d}\; m_i\left(\frac{\pi}{\sqrt{\Lambda}};\Omega\right)\,\Lambda^{\sigma+\frac{d-1}{2}}$$

holds true for $i = 1, \dots, d$ with $\delta_{\sigma,d} = \pi L^{cl}_{\sigma,d} - \varepsilon\left(\sigma + \frac{d-1}{2}\right) L^{cl}_{\sigma,d-1}$.

Remark. Note that in the case of $\varepsilon\left(\sigma + \frac{d-1}{2}\right) = \frac{1}{2}\,B\left(\sigma + \frac{d+1}{2}, \frac{1}{2}\right)$ we have $\delta_{\sigma,d} = 0$. In view of Lemma 4 this occurs if $\sigma + \frac{d-1}{2} \geq 3$, in particular, if $\sigma = \sigma_d$, with σ_d introduced in (12).

Remark. For domians Ω with finite volume (19) yields

$$\int_{\frac{\pi}{\sqrt{\Lambda}}}^{\infty} m_i(\tau;\Omega)d\tau = |\Omega| - M_i\left(\frac{\pi}{\sqrt{\Lambda}};\Omega\right).$$

Thus we arrive at

$$R_\sigma(\Lambda) \;\leq\; L^{cl}_{\sigma,d}\left(|\Omega| - M_i\left(\frac{\pi}{\sqrt{\Lambda}};\Omega\right)\right)\Lambda^{\sigma+\frac{d}{2}} + \delta_{\sigma,d}\, m_i\left(\frac{\pi}{\sqrt{\Lambda}};\Omega\right)\,\Lambda^{\sigma+\frac{d-1}{2}}$$

for $i = 1, \dots, d$. Averaging over all directions one claims

$$R_\sigma(\Lambda) \;\leq\; L^{cl}_{\sigma,d}\left(|\Omega| - M\left(\frac{\pi}{\sqrt{\Lambda}};\Omega\right)\right)\Lambda^{\sigma+\frac{d}{2}} + \delta_{\sigma,d}\, m\left(\frac{\pi}{\sqrt{\Lambda}};\Omega\right)\,\Lambda^{\sigma+\frac{d-1}{2}}, \tag{20}$$

where

$$\begin{aligned} m(t;\Omega) \;&=\; \frac{1}{d}\left(m_1(t;\Omega) + \dots + m_d(t;\Omega)\right), \\ M(y;\Omega) \;&=\; \frac{1}{d}\left(M_1(y;\Omega) + \dots + M_d(y;\Omega)\right) = \int_0^y m(t;\Omega)dt\,. \end{aligned}$$

Although Proposition 5 is, in general, not as sharp as (18), we cannot deduce it directly quoting Proposition 3, but we have to modify the respective proof from [Wei08], which relies on operator-valued Lieb-Thirring inequalities from [LW00].

Proof of Proposition 5. Consider the quadratic form

$$\|\nabla u\|^2_{L^2(\Omega)} - \Lambda\,\|u\|^2_{L^2(\Omega)} \;=\; \|\nabla' u\|^2_{L^2(\Omega)} + \int_{\mathbb{R}^{d-1}} dx' \int_{\Omega(x')} \left(|\partial_{x_d} u|^2 - \Lambda|u|^2\right) dx_d$$

on functions u from the form core $C_0^\infty(\Omega)$. Here ∇' and Δ' denote the gradient and the Laplace operator in the first $d-1$ directions. The functions $u(x',\cdot)$ satisfy Dirichlet boundary conditions at the endpoints of each interval $J_k(x')$ forming $\Omega(x')$. Let the bounded, non-negative operators $W_k(x',\Lambda)$ be the negative parts[4] of

[4]The negative part of a real number r is given by $r_- = (|r| - r)/2 \geq 0$. For operators we use the same convention in the spectral sense.

the Sturm-Liouville Operators $-\partial^2_{x_d,J_k(x')} - \Lambda$ with Dirichlet boundary conditions on $J_k(x')$. Then

$$W(x',\Lambda) = \oplus_{k=1}^{N(x')} W_k(x',\Lambda)$$

is the negative part of

$$-\partial^2_{x_d,\Omega(x')} - \Lambda = \oplus_{k=1}^{N(x')} \left(-\partial^2_{x_d,J_k(x')} - \Lambda\right)$$

subject to Dirichlet boundary conditions on the endpoints of the intervals $J_k(x')$, $k = 1,\dots,N(x')$, that is on $\partial\Omega(x')$. Then

$$\int_{\Omega(x')} \left(|\partial_{x_d} u|^2 - \Lambda |u|^2\right) dx_d \;\geq\; -\langle W u(x',\cdot), u(x',\cdot)\rangle_{L^2(\Omega(x'))}.$$

and consequently

$$\|\nabla u\|^2_{L^2(\Omega)} - \Lambda \|u\|^2_{L^2(\Omega)} \;\geq \|\nabla' u\|^2_{L^2(\Omega)} - \int_{\mathbb{R}^{d-1}} dx' \langle W u(x',\cdot), u(x',\cdot)\rangle_{L^2(\Omega(x'))}. \tag{21}$$

Now we can extend this quadratic form by zero to $C_0^\infty\left(\mathbb{R}^d \setminus \partial\Omega\right)$, which is a form core for $\left(-\Delta_{\mathbb{R}^d\setminus\Omega}\right) \oplus (-\Delta_\Omega - \Lambda)$. This operator corresponds to the left hand side of (21), while the semi-bounded form on the right hand side is closed on the larger domain $H^1\left(\mathbb{R}^{d-1}, L^2(\mathbb{R})\right)$, where it corresponds to the operator

$$-\Delta' \otimes \mathbb{I} - W(x',\Lambda) \quad \text{on} \quad L^2\left(\mathbb{R}^{d-1}, L^2(\mathbb{R})\right). \tag{22}$$

Due to the positivity of $-\Delta_{\mathbb{R}^d\setminus\Omega}$ the variational principle implies that for any $\sigma \geq 0$

$$\begin{aligned} \operatorname{Tr}(-\Delta_\Omega - \Lambda)^\sigma_- \;&=\; \operatorname{Tr}\left(\left(-\Delta_{\mathbb{R}^d\setminus\Omega}\right) \oplus (-\Delta_\Omega - \Lambda)\right)^\sigma_- \\ &\leq\; \operatorname{Tr}(-\Delta' \otimes \mathbb{I} - W(x',\Lambda))^\sigma_- . \end{aligned}$$

We can now apply a sharp Lieb-Thirring inequality to the Schrödinger operator (22) with the operator-valued potential $-W(x',\Lambda)$, see [LW00], and claim that

$$\operatorname{Tr}(-\Delta' \otimes \mathbb{I} - W(x',\Lambda))^\sigma_- \;\leq\; L^{cl}_{\sigma,d-1} \int_{\mathbb{R}^{d-1}} \operatorname{Tr} W^{\sigma+\frac{d-1}{2}}(x',\Lambda)\, dx', \quad \sigma \geq \frac{3}{2}.$$

Now let $p_d(x') = \sum_k |J_k(x')|_1$ be the total length of all intervals $J_k(x')$. Then shifting these intervals and dropping intermediate Dirichlet conditions by a variational argument we see that the j-th eigenvalue of $-\partial^2_{x_d,\Omega(x')} - \Lambda$ is not smaller than the j-th eigenvalue of $-\partial^2_{x_d,L(x')} - \Lambda$ on the interval $L(x') = [0, p_d(x')]$ subject to Dirichlet conditions at the endpoint of this one interval only. Thus,

$$\operatorname{Tr} W^{\sigma+\frac{d-1}{2}}(x',\Lambda) \leq \operatorname{Tr} \tilde{W}^{\sigma+\frac{d-1}{2}}(x',\Lambda),$$

where $\tilde{W}(x',\Lambda)$ is the negative part of $-\partial^2_{x_d,L(x')} - \Lambda$. The nonzero eigenvalues of $\tilde{W}(x',\Lambda)$ are given explicitly by

$$\mu_j = \Lambda - \frac{\pi^2 j^2}{p_d^2(x')} = \Lambda\left(1 - \frac{l_\Lambda^2 j^2}{p_d^2(x')}\right) \quad \text{for} \quad j = 1,\dots,\left[\frac{p_d(x')}{l_\Lambda}\right].$$

From this we conclude that

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)^\sigma_- \;\leq\; \Lambda^{\sigma+\frac{d-1}{2}} L^{cl}_{\sigma,d-1} \int_{\mathbb{R}^{d-1}} \sum_{j\geq 1} \left(1 - \frac{l_\Lambda^2 j^2}{p_d^2(x')}\right)_+^{\sigma+\frac{d-1}{2}} dx'.$$

Note that the right hand side of this bound vanishes if $p_d(x') \le l_\Lambda$. For $p_d(x') > l_\Lambda$ we have in view of (17)

$$\sum_{j\ge 1}\left(1-\frac{l_\Lambda^2 j^2}{p_d^2(x')}\right)_+^{\sigma+\frac{d-1}{2}} \le \frac{p_d(x')}{2\,l_\Lambda}\,B\left(\sigma+\frac{d+1}{2},\frac{1}{2}\right)-\varepsilon\left(\sigma+\frac{d-1}{2}\right)$$

and therefore

$$\begin{aligned}\operatorname{Tr}(-\Delta_\Omega-\Lambda)_-^\sigma &\le \frac{1}{2\pi}\,B\left(\sigma+\frac{d+1}{2},\frac{1}{2}\right)\Lambda^{\sigma+\frac{d}{2}}L^{cl}_{\sigma,d-1}\int_{x':p_d(x')>l_\Lambda} p_d(x')\,dx' \\ &\quad -\varepsilon\left(\sigma+\frac{d-1}{2}\right)\Lambda^{\sigma+\frac{d-1}{2}}\,L^{cl}_{\sigma,d-1}\int_{x':p_d(x')>l_\Lambda} dx'. \qquad (23)\end{aligned}$$

Note that

$$\int_{x':p_d(x')>l_\Lambda} dx' = m_d\,(l_\Lambda;\Omega)$$

and

$$\int_{x':p_d(x')>l_\Lambda} p_d(x')\,dx' = m_d\,(l_\Lambda;\Omega)\,l_\Lambda+\int_{l_\Lambda}^\infty m_d\,(\tau;\Omega)\,d\tau\,.$$

Moreover, using

$$\frac{1}{2\pi}B\left(\sigma+\frac{d+1}{2},\frac{1}{2}\right)L^{cl}_{\sigma,d-1}=L^{cl}_{\sigma,d},$$

we insert the identities above into (23) and arrive at

$$\begin{aligned}R_\sigma(\Lambda) = \operatorname{Tr}(-\Delta-\Lambda)_-^\sigma &\le L^{cl}_{\sigma,d}\,\Lambda^{\sigma+\frac{d}{2}}\left(m_d\,(l_\Lambda;\Omega)\,l_\Lambda+\int_{l_\Lambda} m_d(\tau;\Omega)d\tau\right) \\ &\quad -\varepsilon\left(\sigma+\frac{d-1}{2}\right)L^{cl}_{\sigma,d-1}\,m_d\,(l_\Lambda;\Omega)\,\Lambda^{\sigma+\frac{d-1}{2}}\,.\end{aligned}$$

In view of $l_\Lambda=\pi\Lambda^{-1/2}$ this yields

$$R_\sigma(\Lambda)\le L^{cl}_{\sigma,d}\int_{\frac{\pi}{\sqrt{\Lambda}}} m_d(\tau;\Omega)d\tau\,\Lambda^{\sigma+\frac{d}{2}}+\delta_{\sigma,d}\,m_d\,(l_\Lambda;\Omega)\,\Lambda^{\sigma+\frac{d-1}{2}},\quad \sigma\ge\frac{3}{2}.$$

Interchanging the roles of x_d and x_i we find the respective inequalities for any direction $i=1,\dots,d$. □

In order to derive universal bounds on R_σ independent from M, in particular to prove Theorem 2, one needs bounds on $M(y)$. Identity (19) immediately implies

$$M(y;\Omega)=\int_0^y m(\tau;\Omega)d\tau\le\int_0^\infty m(\tau;\Omega)d\tau=|\Omega|\quad\text{for all}\quad 0<y<\infty\,. \qquad (24)$$

To prove a lower bound we first need an auxiliary result concerning rearrangements of Ω. For $\Omega\subset\mathbb{R}^d$, $d\ge 2$, fix a Cartesian coordinate system $(x',x_d)\in\mathbb{R}^{d-1}\times\mathbb{R}$. Again put

$$p_d(x';\Omega)=|\{x_d:(x',x_d)\in\Omega\}|_1=|\Omega(x')|_1$$

and for $\tau>0$

$$\Omega^*(\tau)=\{x':p_d(x';\Omega)>\tau\}\subset\mathbb{R}^{d-1}.$$

This is a non-increasing set function, that means $\Omega^*(\tau_1)\supset\Omega^*(\tau_2)$ for $0<\tau_1\le\tau_2$. Let

$$\Omega^*=\cup_{\tau>0}\,\Omega^*(\tau)\times\{\tau\}\subset\mathbb{R}^d \qquad (25)$$

be a non-increasing rearrangement of Ω in the direction of the x_d-coordinate. Then we have

Lemma 6. *For all $i = 1, \ldots, d$ and all $y > 0$*

$$M_i(y;\Omega) \;\geq\; M_i(y;\Omega^*).$$

Proof. First note that in the case $i = d$ we have by construction $p_d(x';\Omega) = p_d(x';\Omega^*)$ and consequently

$$m_d(\tau;\Omega) = m_d(\tau;\Omega^*) = |\Omega^*(\tau)|_{d-1}\,,$$

what implies $M_d(y;\Omega) = M_d(y;\Omega^*)$.

Assume now that $j = 1, \ldots, d-1$. Put

$$x'' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{d-1}) \in \mathbb{R}^{d-2}$$

and

$$p_j(x'', x_d;\Omega) = |\{x_j : (x', x_d) \in \Omega\}|_1\,.$$

By definition

$$m_j(s;\Omega) = |\{(x'', x_d) : p_j(x'', x_d;\Omega) > s\}|_{d-1} = \int_{\mathbb{R}^{d-2}} \hat{m}_j(x'', s;\Omega)\, dx''$$

where

$$\hat{m}_j(x'', s;\Omega) = |\{x_d : p_j(x'', x_d;\Omega) > s\}|_1\,, \quad j = 1, \ldots, d-1\,.$$

Hence,

$$M_j(y;\Omega) = \int_0^y m_j(s;\Omega) ds = \int_{\mathbb{R}^{d-2}} \int_0^y \hat{m}_j(x'', s;\Omega)\, ds\, dx''.$$

Applying the same notation to Ω^* yields

$$M_j(y;\Omega^*) = \int_0^y m_j(s;\Omega^*)\, ds = \int_{\mathbb{R}^{d-2}} \int_0^y \hat{m}_j(x'', s;\Omega^*)\, ds\, dx''.$$

If we can show that for $x'' \in \mathbb{R}^{d-2}$ and all $y > 0$ the inequality

$$\int_0^y \hat{m}_j(x'', s;\Omega) ds \geq \int_0^y \hat{m}_j(x'', s;\Omega^*) ds \tag{26}$$

holds true, the assertion is proven.

To establish (26) we consider for fixed $x'' \in \mathbb{R}^{d-2}$ the two-dimensional sets

$$\hat{\Omega} \;=\; \{(x_j, x_d) : (x', x_d) \in \Omega\} \quad \text{and} \quad \hat{\Omega}^* \;=\; \{(x_j, x_d) : (x', x_d) \in \Omega^*\}\;.$$

Note that

$$p_d(x';\Omega) = |\{x_d : (x', x_d) \in \Omega\}|_1 = \left|\{x_d : (x_j, x_d) \in \hat{\Omega}\}\right|_1 =: \hat{p}_d(x_j;\hat{\Omega})\,.$$

As above we get

$$\hat{p}_d(x_j;\hat{\Omega}) = \hat{p}_d(x_j;\hat{\Omega}^*)\,. \tag{27}$$

In the jth direction we have

$$p_j(x'', x_d;\Omega) = |\{x_j : (x', x_d) \in \Omega\}|_1 = \left|\left\{x_j : (x_j, x_d) \in \hat{\Omega}\right\}\right|_1 =: \hat{p}_j(x_d;\hat{\Omega})$$

and

$$\hat{m}_j(x'', s;\Omega) = |\{x_d : p_j(x'', x_d;\Omega) > s\}|_1 = \left|\left\{x_d : \hat{p}_j(x_d, \hat{\Omega}) > s\right\}\right|_1 =: \hat{m}_j(s;\hat{\Omega})\,.$$

The corresponding notions we use also with respect to the domains Ω^* and $\hat{\Omega}^*$. In contrast to the preservation of length in the dth direction the values of $\hat{p}_j(x_d;\hat{\Omega})$ and $\hat{p}_j(x_d;\hat{\Omega}^*)$ (and thus of $\hat{m}_j(s;\hat{\Omega})$ and $\hat{m}_j(s;\hat{\Omega}^*)$) do not coincide in general.

Lets examine the functions $\hat{p}_j(x_d;\hat{\Omega}^*)$ and $\hat{m}_j(s;\hat{\Omega}^*)$ in more detail. By construction of $\hat{\Omega}^*$, the set function $\hat{\Omega}^*(x_d) = \{x_j : (x_j,x_d) \in \hat{\Omega}\}$ is non-increasing in $x_d > 0$ and by definition

$$\hat{p}_j\left(x_d;\hat{\Omega}^*\right) = \left|\hat{\Omega}^*(x_d)\right|_1 .$$

Moreover, $\hat{m}_j(s;\hat{\Omega}^*)$ is the distribution function of $\hat{p}_j(x_d;\hat{\Omega}^*)$. Hence,

$$\int_0^y \hat{m}_j(s;\hat{\Omega}^*)\,ds = \int_{\left\{x_d:\hat{p}_j(x_d;\hat{\Omega}^*)<y\right\}} \hat{p}_j(x_d;\hat{\Omega}^*)\,dx_d + y\,\left|\left\{x_d : \hat{p}_j(x_d;\hat{\Omega}^*) \geq y\right\}\right|_1 .$$

The monotonicity of the set function $\hat{\Omega}^*(x_d)$ implies, that we can choose $I_y \subset \mathbb{R}$ with total length y satisfying $I_y \subset \hat{\Omega}^*(x_d)$, wherever $\hat{p}_j(x_d;\hat{\Omega}^*) \geq y$. Again, by the monotonicity of $\hat{\Omega}^*(x_d)$ the reverse inclusion $\hat{\Omega}^*(x_d) \subset I_y$ holds for all $x_d > 0$ with $\hat{p}_j(x_d;\hat{\Omega}^*) < y$. Put

$$\hat{\Omega}^*_y = \bigcup_{x_d>0}\left(\hat{\Omega}^*(x_d)\cap I_y\right)\times\{x_d\} \quad \text{and} \quad \hat{\Omega}_y = \bigcup_{x_d>0}\left(\hat{\Omega}(x_d)\cap I_y\right)\times\{x_d\}.$$

From the above representation for $\int_0^y \hat{m}_j(s;\hat{\Omega}^*)\,ds$ we deduce

$$\int_0^y \hat{m}_j(s;\hat{\Omega}^*)ds = \left|\hat{\Omega}^*_y\right| . \tag{28}$$

Moreover, note that for $x_j \in I_y$

$$\begin{aligned}\left\{x_d : (x_j,x_d) \in \hat{\Omega}^*_y\right\} &= \left\{x_d : (x_j,x_d) \in \hat{\Omega}^*\right\} \quad \text{and}\\ \left\{x_d : (x_j,x_d) \in \hat{\Omega}_y\right\} &= \left\{x_d : (x_j,x_d) \in \hat{\Omega}\right\}.\end{aligned}$$

In view of (27) we get

$$\hat{p}_d(x_j;\hat{\Omega}_y) = \hat{p}_d(x_j;\hat{\Omega}) = \hat{p}_d(x_j;\hat{\Omega}^*) = \hat{p}_d(x_j;\hat{\Omega}^*_y)$$

and we conclude that

$$\left|\hat{\Omega}^*_y\right| = \left|\hat{\Omega}_y\right| . \tag{29}$$

Finally, we analyse $\hat{m}_j(s,\hat{\Omega})$. The inclusion $\hat{\Omega}_y \subset \hat{\Omega}$ implies

$$\int_0^y \hat{m}_j(s;\hat{\Omega})\,ds \geq \int_0^y \hat{m}_j(s;\hat{\Omega}_y)\,ds . \tag{30}$$

Moreover, by construction of $\hat{\Omega}_y$ we have

$$\hat{p}_j(x_d;\hat{\Omega}_y) \leq |I_y| = y$$

for all $x_d > 0$ and consequently $m_j(s;\hat{\Omega}_y) = 0$ for all $s \geq y$. Using (30) we conclude

$$\int_0^y \hat{m}_j(s;\hat{\Omega})\,ds \geq \int_0^y \hat{m}_j(s;\hat{\Omega}_y)\,ds = \int_0^\infty \hat{m}_j(s;\hat{\Omega}_y)\,ds = \left|\hat{\Omega}_y\right| .$$

In view (29) and (28) we arrive at

$$\int_0^y \hat{m}_j(s;\hat{\Omega})\,ds \geq \left|\hat{\Omega}_y\right| = \left|\hat{\Omega}^*_y\right| = \int_0^y \hat{m}_j(s;\hat{\Omega}^*)\,ds .$$

This shows that (26) holds true and the proof is complete. □

Now we can give a lower bound on $M(y;\Omega)$:

Lemma 7. *For all open sets $\Omega \subset \mathbb{R}^d$ and all $y > 0$*

$$M(y;\Omega) \ \geq\ \min\left(\frac{|\Omega|}{d},\, |\Omega|^{\frac{d-1}{d}}\, y\right). \tag{31}$$

Proof. We use induction in the dimension. For $d = 1$ and an interval of length $|\Omega|$ we get

$$m(\tau;\Omega) \ =\ \begin{cases} 1 & |\Omega| > \tau \\ 0 & |\Omega| \leq \tau \end{cases}$$

and therefore $M(y;\Omega) = \int_0^y m(\tau;\Omega)d\tau = \min(y, |\Omega|)$ for all $y > 0$.

Now assume $\Omega \subset \mathbb{R}^d$, $d \geq 2$. For any given $j = 1, \ldots, d-1$ put

$$x'' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{d-1}) \in \mathbb{R}^{d-2}$$

and let $\tilde{m}_j(s;\tilde{\Omega}) = |\{x'' : \tilde{p}_j(x'';\tilde{\Omega}) > s\}|_{d-2}$ be the distribution function of a set $\tilde{\Omega} \subset \mathbb{R}^{d-1}$ with respect to the j-th direction, where $\tilde{p}_j(x'';\tilde{\Omega}) = |\{x_j : x' \in \tilde{\Omega}\}|_1$ is the total length of the section through $\tilde{\Omega}$ at x'' in the direction of the x_j-coordinate. Applying these notions to Ω^* given in (25) we get

$$\begin{aligned} m_j(s;\Omega^*) &= \left|\{(x'',\tau) \in \mathbb{R}^{d-1} : p_j\left(x'',\tau;\Omega^*\right) > s\}\right|_{d-1} \\ &= \int_0^\infty |\{x'' \in \mathbb{R}^{d-2} : \tilde{p}_j\left(x'';\Omega^*(\tau)\right) > s\}|_{d-2}\, d\tau \\ &= \int_0^\infty \tilde{m}_j(s;\Omega^*(\tau))\, d\tau\,, \qquad j = 1, \ldots, d-1\,. \end{aligned} \tag{32}$$

Put $\tilde{m}(s;\tilde{\Omega}) = (d-1)^{-1}\sum_{j=1}^{d-1}\tilde{m}_j(s;\tilde{\Omega})$. By induction assumption we have

$$\tilde{M}(y;\tilde{\Omega}) = \int_0^y \tilde{m}(s;\tilde{\Omega})\, ds \geq \min\left(\frac{|\tilde{\Omega}|_{d-1}}{d-1},\, |\tilde{\Omega}|_{d-1}^{\frac{d-2}{d-1}}\, y\right), \quad y > 0. \tag{33}$$

Next note that in view of (32)

$$\begin{aligned} d \cdot M(y;\Omega^*) &= M_1(y;\Omega^*) + \cdots + M_{d-1}(y;\Omega^*) + M_d(y;\Omega^*) \\ &= \int_0^y (m_1(s;\Omega^*) + \cdots + m_{d-1}(s;\Omega^*))\, ds + \int_0^y m_d(s;\Omega^*)\, ds \\ &= (d-1)\int_0^y\int_0^\infty \tilde{m}(s;\Omega^*(\tau))\, d\tau\, ds + \int_0^y m_d(s;\Omega^*)\, ds \\ &= (d-1)\int_0^\infty \tilde{M}(y;\Omega^*(\tau))\, d\tau + \int_0^y m_d(s;\Omega^*)\, ds\,. \end{aligned}$$

Using (33) we claim

$$M(y;\Omega^*) \geq \frac{d-1}{d}\int_0^\infty \min\left(\frac{|\Omega^*(\tau)|_{d-1}}{d-1},\, |\Omega^*(\tau)|_{d-1}^{\frac{d-2}{d-1}}\, y\right) d\tau + \frac{1}{d}\int_0^y m_d(s;\Omega^*)ds\,.$$

We point out that $|\Omega^*(\tau)|_{d-1} = m_d(\tau,\Omega^*)$ for $\tau > 0$. Put

$$\tau^* = \inf\left\{\tau > 0 : m_d(\tau;\Omega^*) \leq (d-1)^{d-1}y^{d-1}\right\}\,.$$

Then

$$M(y;\Omega^*) \ge \frac{1}{d}\int_{\tau^*}^{\infty} m_d\,(\tau,\Omega^*)\,d\tau \quad + \quad \frac{1}{d}\int_0^y m_d(\tau;\Omega^*)d\tau \\ \quad + \quad \frac{d-1}{d} y \int_0^{\tau^*} m_d^{\frac{d-2}{d-1}}\,(\tau;\Omega^*)\,d\tau\,.$$

By (19) we have $\int_0^\infty m_d\,(\tau;\Omega^*)\,d\tau = \int_0^\infty m_d\,(\tau;\Omega)\,d\tau = |\Omega|$ and using Lemma 6 we estimate

$$\begin{aligned} M(y;\Omega) \ \ge\ M(y;\Omega^*) \quad &\ge \quad \frac{|\Omega|}{d} - \frac{1}{d}\int_0^{\tau^*} m_d\,(\tau;\Omega^*)\,d\tau + \frac{1}{d}\int_0^y m_d(\tau;\Omega^*)d\tau \\ &\quad + \frac{d-1}{d} y \int_0^{\tau^*} m_d^{\frac{d-2}{d-1}}\,(\tau;\Omega^*)\,d\tau\,. \end{aligned} \tag{34}$$

In particular, in the case of $\tau^* \le y$ we see from the previous bound that

$$M(y;\Omega) \ge d^{-1}\,|\Omega|$$

and the assertion is proven. Hence, let us consider the remaining case $\tau^* > y$ in more detail. For $\tau^* > y$ we have

$$m_d(y;\Omega^*) \ge (d-1)^{d-1} y^{d-1}\,.$$

Because of the monotonicity of m_d we conclude that

$$\int_0^y m_d(\tau;\Omega^*)^{\frac{d-2}{d-1}} d\tau \ \ge\ y\, m_d^{\frac{d-2}{d-1}}(y;\Omega^*) \quad \text{and} \quad \int_0^y m_d(\tau;\Omega^*)d\tau \ \ge\ y\, m_d(y;\Omega^*)\,.$$

Let us rewrite inequality (34) as follows

$$\begin{aligned} M(y;\Omega) \ge \frac{|\Omega|}{d} \quad &+ \quad \frac{d-1}{d}\, y \int_0^y m_d^{\frac{d-2}{d-1}}(\tau;\Omega^*)d\tau \\ &+ \quad \frac{d-1}{d}\, y \int_y^{\tau^*} m_d^{\frac{d-2}{d-1}}(\tau;\Omega^*)d\tau - \frac{1}{d}\int_y^{\tau^*} m_d(\tau;\Omega^*)d\tau\,. \end{aligned}$$

Put $A = \int_y^{\tau^*} m_d\,(\tau;\Omega^*)\,d\tau$. Then

$$0 < A = \int_0^{\tau^*} m_d\,(\tau;\Omega^*)\,d\tau - \int_0^y m_d\,(\tau;\Omega^*)\,d\tau \le |\Omega| - y\, m_d(y;\Omega^*)\,. \tag{35}$$

Moreover,

$$M(y;\Omega) \ge \frac{|\Omega|}{d} + \frac{d-1}{d}\, y^2\, m_d^{\frac{d-2}{d-1}}(y;\Omega^*) + \frac{d-1}{d}\, y \int_y^{\tau^*} m_d^{\frac{d-2}{d-1}}(\tau;\Omega^*)d\tau - \frac{A}{d}\,.$$

Due to the monotonicity of m_d we have, in particular, $m_d(\tau;\Omega^*) \le m_d(y;\Omega^*)$ for $y \le \tau$ and

$$\begin{aligned} \int_y^{\tau^*} m_d^{\frac{d-2}{d-1}}(\tau;\Omega^*)d\tau \quad &= \quad m_d^{\frac{d-2}{d-1}}(y;\Omega^*) \int_y^{\tau^*} \left(\frac{m_d(\tau;\Omega^*)}{m_d(y;\Omega^*)}\right)^{\frac{d-2}{d-1}} d\tau \\ &\ge \quad m_d^{\frac{d-2}{d-1}}(y;\Omega^*) \int_y^{\tau^*} \frac{m_d(\tau;\Omega^*)}{m_d(y;\Omega^*)}\, d\tau = m_d^{\frac{-1}{d-1}}(y;\Omega^*)\, A\,. \end{aligned}$$

Thus,

$$M(y;\Omega) \ge \frac{|\Omega|}{d} + \frac{d-1}{d}\, y^2\, m_d^{\frac{d-2}{d-1}}(y;\Omega^*) - \frac{1}{d}\left(1 - (d-1)\, y\, m_d^{\frac{-1}{d-1}}(y;\Omega^*)\right) A\,.$$

For $\tau^* > y$ we have $1-(d-1)\,y\,m_d^{\frac{-1}{d-1}}(y;\Omega^*) > 0$ and we can insert (35) in this estimate and arrive at

$$\begin{aligned} M(y;\Omega) &\geq \frac{|\Omega|}{d} + \frac{d-1}{d}\,y^2\,m_d^{\frac{d-2}{d-1}}(y;\Omega^*) \\ &\quad -\frac{1}{d}\left(1-(d-1)\,y\,m_d^{\frac{-1}{d-1}}(y;\Omega^*)\right)(|\Omega| - y\,m_d(y;\Omega^*)) \\ &\geq \frac{y}{d}\left((d-1)|\Omega|\,m_d^{\frac{-1}{d-1}}(y;\Omega^*) + m_d(y;\Omega^*)\right). \end{aligned}$$

Since the function $f(m) = (d-1)|\Omega|\,m^{\frac{-1}{d-1}} + m$ takes its minimal value for positive arguments at $m = |\Omega|^{\frac{d-1}{d}}$, we arrive for $y < \tau^*$ at

$$M(y;\Omega) \geq \frac{y}{d} f(m_d(y;\Omega^*)) \geq \frac{y}{d} f(|\Omega|^{\frac{d-1}{d}}) = y\,|\Omega|^{\frac{d-1}{d}}.$$

This completes the proof. $\square$

4. Proof of Theorem 1 and remarks

Let

$$\mathcal{L}[f(\cdot)](t) = \int_0^\infty f(\Lambda)e^{-\Lambda t}d\Lambda$$

be the Laplace transformation of a suitable function $f:(0,+\infty)\to\mathbb{R}$. For real values of t it is monotone, that means a pointwise estimate $f_1(\Lambda)\leq f_2(\Lambda)$ for all $\Lambda > 0$ implies $\mathcal{L}[f_1](t)\leq\mathcal{L}[f_2](t)$ for any $t\in\mathbb{R}$, for which both transformations are defined. In particular, for $\lambda\geq 0$ and $\sigma > 0$ one has

$$\mathcal{L}[(\Lambda-\lambda)_+^\sigma](t) = \int_\lambda^\infty (\Lambda-\lambda)^\sigma e^{-\Lambda t}d\Lambda = e^{-\lambda t}t^{-\sigma-1}\Gamma(\sigma+1)\,,\quad t>0\,.$$

In view of the linearity of the Laplace transformation one finds for $t>0$ and $\sigma>0$ the well-known identity

$$Z(t) = \operatorname{Tr} e^{+\Delta_\Omega t} = \sum_k e^{-\lambda_k t} = \sum_k \frac{t^{\sigma+1}}{\Gamma(\sigma+1)}\mathcal{L}[(\Lambda-\lambda_k)_+^\sigma](t) = \frac{t^{\sigma+1}}{\Gamma(\sigma+1)}\mathcal{L}[R_\sigma(\Lambda)]\,.$$

Therefore, any bound on the Riesz means of the type

$$R_\sigma(\Lambda)\leq f(\Lambda,\Omega)\quad\text{for all}\quad\Lambda>0 \tag{36}$$

implies a bound on the heat kernel

$$Z(t)\leq\frac{t^{\sigma+1}}{\Gamma(\sigma+1)}\mathcal{L}[f(\cdot,\Omega)](t) \tag{37}$$

valid for all $t>0$, for which the r.h.s. is defined. For example, this way one can deduce (1) from (3) with any $\sigma\geq 1$.

Next note that in view of $R_\sigma(\Lambda)=0$ for $0<\Lambda\leq\lambda_1$ we have in fact

$$\Gamma(\sigma+1)t^{-\sigma-1}Z(t) = \mathcal{L}[R_\sigma](t) = \mathcal{L}[R_\sigma,\lambda](t)\quad\text{for any}\quad 0\leq\lambda\leq\lambda_1\,,$$

where

$$\mathcal{L}[f,\lambda](t) = \int_\lambda^\infty f(\Lambda)e^{-\Lambda t}d\Lambda = e^{-\lambda t}\mathcal{L}[f(\cdot+\lambda)](t)\,,\quad\lambda\geq 0\,,$$

is the reduced Laplace transformation of a suitable function f. This transformation preserves pointwise inequalities as well and from (36) one can deduce an improved version of (37)

$$Z(t) \le \frac{t^{\sigma+1}}{\Gamma(\sigma+1)} \mathcal{L}[f(\cdot,\Omega),\lambda](t) \quad \text{for arbitrary} \quad 0 \le \lambda \le \lambda_1 \, .$$

Applying this bound to (3) one gets the estimate

$$Z(t) \le \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \hat{\Gamma}\left(\sigma + \frac{d}{2} + 1, \lambda t\right), \quad t > 0\,,\ \sigma \ge 1\,,\ 0 \le \lambda \le \lambda_1\,, \tag{38}$$

which already contains an exponential decay for large t. Instead of referring to the classical Berezin-Li-Yau-bound (3) we can apply this idea also directly to the improved bound (18) and claim

$$\begin{aligned} Z(t) \quad \le \quad & \frac{t^{\sigma+1}}{\Gamma(\sigma+1)} L^{cl}_{\sigma,d} \int_{\lambda_1}^{\infty} |\Omega_\Lambda| \, \Lambda^{\sigma+\frac{d}{2}} e^{-\Lambda t} \, d\Lambda \\ & - \frac{t^{\sigma+1}}{\Gamma(\sigma+1)} L^{cl}_{\sigma,d-1} \, \varepsilon\left(\sigma + \tfrac{d-1}{2}\right) \int_{\lambda_1}^{\infty} d_\Lambda(\Omega) \Lambda^{\sigma+\frac{d-1}{2}} e^{-\Lambda t} \, d\Lambda, \end{aligned} \tag{39}$$

where $t > 0$ and $\sigma \ge \frac{3}{2}$. This bound is even sharper than the estimates presented below. But the geometric properties of Ω enter in a rather tricky way and cannot be simplified in a straightforward manner. Therefore we prefer to present also a slightly weaker, but sometimes more convenient version of this bound. For that end we choose $\sigma = \sigma_d$ given in (12) and apply the reduced Laplace transformation to (20). Thus we get the following estimate valid for $\lambda \in [\tilde{\lambda}, \lambda_1]$ and $t > 0$:

$$\begin{aligned} Z(t) \quad \le \quad & \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \hat{\Gamma}\left(\sigma_d + \frac{d}{2} + 1, \lambda t\right) \\ & - \frac{t^{\sigma_d+1}}{\Gamma(\sigma_d+1)} L^{cl}_{\sigma_d,d} \int_{\lambda}^{\infty} M\left(\frac{\pi}{\sqrt{\Lambda}}\right) \Lambda^{\sigma_d+\frac{d}{2}} e^{-\Lambda t} d\Lambda \, . \end{aligned} \tag{40}$$

We are now in the position to provide bounds on $Z(t)$ depending only on the volume of Ω. To this end we use inequality (7) and calculate $M\left(\frac{\pi}{\sqrt{\Lambda}}\right)$ explicitly on the ball.

Proposition 8. *Let $\lambda \in [\tilde{\lambda}, \lambda_1]$. For any open set $\Omega \subset \mathbb{R}^d$ and any $t > 0$ the bound*

$$\begin{aligned} Z(t) \ \le & \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \, \hat{\Gamma}\left(\sigma_d + \frac{d}{2} + 1, \lambda t\right) \\ & - \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}} \, \Gamma\left(\sigma_d + \frac{d}{2} + 1\right)} \int_{\lambda t}^{\infty} e^{-s} s^{\sigma_d + \frac{d}{2}} \tilde{B}\left(\frac{\pi^2 t}{4R^2 s}, \frac{1}{2}, \frac{d+1}{2}\right) ds \end{aligned}$$

holds true, where $R = R(|\Omega|)$ is the radius of the ball $B_R \subset \mathbb{R}^d$ with $|B_R| = |\Omega|$.

Proof. Lets consider the ball B_R and apply (40) to estimate $Z^*(t)$, i.e. $Z(t)$ on B_R. Note that $m_i(\tau; B_R) = m(\tau; B_R)$ for $i = 1, \ldots, d$ and we can choose an arbitrary coordinate system $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$.

Again put $p_d(x'; B_R) = |\{x_d : (x', x_d) \in B_R\}|$ and note that for $\tau < 2R$ the set $\left\{x' \in \mathbb{R}^{d-1} : p_d(x', B_R) > \tau\right\}$ is itself a ball in $\mathbb{R}^{d-1}$ with radius $\left(R^2 - \tau^2/4\right)^{\frac{1}{2}}$.

Thus we find

$$m(\tau; B_R) = |\{x' : p_d(x', B_R) > \tau\}|_{d-1} \;=\; \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} R^{d-1} \left(1 - \frac{\tau^2}{4R^2}\right)_+^{\frac{d-1}{2}}$$

and

$$M(y; B_R) = \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} R^d\, B\left(0, \frac{y^2}{4R^2}, \frac{1}{2}, \frac{d+1}{2}\right) = |B_R|\; \tilde{B}\left(\frac{y^2}{4R^2}, \frac{1}{2}, \frac{d+1}{2}\right).$$

We insert this estimate into (40) and arrive at

$$\begin{aligned} Z^*(t) \;\leq\; & \frac{|B_R|}{(4\pi t)^{\frac{d}{2}}} \hat{\Gamma}\left(\sigma_d + \frac{d}{2} + 1, \lambda t\right) \\ & - \frac{t^{\sigma_d+1}}{\Gamma(\sigma_d+1)} L^{cl}_{\sigma_d,d}\, |B_R| \int_\lambda^\infty \tilde{B}\left(\frac{\pi^2}{4R^2\Lambda}, \frac{1}{2}, \frac{d+1}{2}\right) \Lambda^{\sigma_d+\frac{d}{2}} e^{-\Lambda t} d\Lambda. \end{aligned}$$

The assumption $\lambda \geq \tilde{\lambda}$ implies $\frac{\pi^2}{4R^2} < \lambda$ and in view of (7) and $|B_R| = |\Omega|$ the claimed result follows by simplifying the right hand side. $\square$

We can now derive Theorem 1 from Proposition 8:

Proof of Theorem 1. The inequality

$$(1-u)^{\frac{d-1}{2}} \geq 1 - \frac{d-1}{2} u\,, \quad 0 \leq u \leq 1\,,$$

implies the estimate

$$\begin{aligned} \tilde{B}\left(\frac{\pi^2 t}{4R^2 s}, \frac{1}{2}, \frac{d+1}{2}\right) &\geq \frac{1}{B\left(\frac{1}{2}, \frac{d+1}{2}\right)} \int_0^{\frac{\pi^2 t}{4R^2 s}} u^{-\frac{1}{2}} \left(1 - \frac{d-1}{2} u\right) du \\ &= \frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d+1}{2}\right)} \left(\frac{\sqrt{\pi t}}{R\sqrt{s}} - \frac{(d-1)\pi^{\frac{5}{2}} t^{\frac{3}{2}}}{24 R^3 s^{\frac{3}{2}}}\right). \end{aligned}$$

Therefore we claim

$$\begin{aligned} & \int_{\lambda t}^\infty e^{-s} s^{\sigma_d+\frac{d}{2}} \tilde{B}\left(\frac{\pi^2\, t}{4R^2\, s}, \frac{1}{2}, \frac{d+1}{2}\right) ds \\ & \geq \frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d+1}{2}\right)} \left(\frac{\sqrt{\pi t}}{R} \Gamma\left(\sigma_d + \frac{d+1}{2}, \lambda t\right) - \frac{(d-1)\pi^{\frac{5}{2}} t^{\frac{3}{2}}}{24R^3} \Gamma\left(\sigma_d + \frac{d-1}{2}, \lambda t\right)\right). \end{aligned}$$

Inserting the last estimate into the bound from Proposition 8 yields

$$Z(t) \leq \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \hat{\Gamma}\left(\sigma_d + \frac{d}{2} + 1, \lambda t\right) - R(t, \lambda)$$

with $R(t,\lambda) = r_1(t,\lambda) - r_2(t,\lambda)$ and

$$\begin{aligned} r_1(t,\lambda) &= \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d+1}{2}\right)} \frac{\sqrt{\pi t}}{R} \frac{\Gamma\left(\sigma_d + \frac{d+1}{2}, \lambda t\right)}{\Gamma\left(\sigma_d + \frac{d}{2} + 1\right)} \\ r_2(t,\lambda) &= \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d+1}{2}\right)} \frac{(d-1)\pi^{\frac{5}{2}} t^{\frac{3}{2}}}{24R^3} \frac{\Gamma\left(\sigma_d + \frac{d-1}{2}, \lambda t\right)}{\Gamma\left(\sigma_d + \frac{d}{2} + 1\right)}. \end{aligned}$$

From $|B_R| = |\Omega|$ we deduce

$$R = \frac{|\Omega|^{\frac{1}{d}}}{\sqrt{\pi}} \Gamma\left(\frac{d}{2}+1\right)^{\frac{1}{d}}, \tag{41}$$

and get

$$r_1(t,\lambda) = \frac{|\Omega|^{\frac{d-1}{d}}}{(4\pi t)^{\frac{d-1}{2}}} \frac{B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right)}{2} \frac{\Gamma\left(\frac{d}{2}+1\right)^{\frac{d-1}{d}}}{\Gamma\left(\frac{d+1}{2}\right)} \hat{\Gamma}\left(\sigma_d + \frac{d+1}{2}, \lambda t\right)$$

$$r_2(t,\lambda) = \frac{|\Omega|^{\frac{d-3}{d}}}{(4\pi t)^{\frac{d-3}{2}}} \frac{\pi^2(d-1) B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right)}{96(2\sigma_d + d - 1)} \frac{\Gamma\left(\frac{d}{2}+1\right)^{\frac{d-3}{d}}}{\Gamma\left(\frac{d+1}{2}\right)} \hat{\Gamma}\left(\sigma_d + \frac{d-1}{2}, \lambda t\right).$$

To complete the proof it remains to note that in view of (38) we can always estimate the remainder term $R(t,\lambda)$ from above by zero. □

Remark. According to (11) we can choose

$$\tilde{\lambda} = \frac{\pi\, j^2_{\frac{d}{2}-1,1}}{\Gamma\left(\frac{d}{2}+1\right)^{2/d} |\Omega|^{2/d}}$$

as a suitable lower bound on λ_1. With this special choice of parameter we find

Corollary 9. *For any open set $\Omega \subset \mathbb{R}^d$ with finite volume and all $t > 0$*

$$Z(t) \le \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \hat{\Gamma}\left(\sigma_d + \frac{d}{2} + 1, \tilde{\lambda}\, t\right) - (R(t))_+ \tag{42}$$

holds true with

$$R(t) = c_{1,d} \frac{|\Omega|^{\frac{d-1}{d}}}{(4\pi t)^{\frac{d-1}{2}}} \hat{\Gamma}\left(\sigma_d + \frac{d+1}{2}, \tilde{\lambda} t\right) - c_{2,d} \frac{|\Omega|^{\frac{d-3}{d}}}{(4\pi t)^{\frac{d-3}{2}}} \hat{\Gamma}\left(\sigma_d + \frac{d-1}{2}, \tilde{\lambda} t\right)$$

and constants $c_{1,d}$, $c_{2,d}$ given explicitly in Theorem 1.

Finally, we can apply (7) to known estimates on $Z(t)$ and compare the resulting universal bounds with the result from Corollary 9.

To analyse the asymptotics of $Z(t)$ for $t \to 0+$ on convex domains van den Berg proved [vdB84b] that for all convex domains $D \subset \mathbb{R}^d$ and all $t > 0$

$$Z(t) \le \frac{|D|}{(4\pi t)^{\frac{d}{2}}} - \frac{|\partial D|}{4(4\pi t)^{\frac{d-1}{2}}} + \frac{(d-1)\,|\partial D|\, t}{(4\pi t)^{\frac{d}{2}} 2R},$$

where ∂D denotes the boundary of D and at each point of ∂D the curvature is bounded by $\frac{1}{R}$. To prove bounds for general domains Ω we can apply this bound to the ball. Note that

$$|\partial B_R| = d\,\pi^{\frac{d}{2}} \frac{R^{d-1}}{\Gamma\left(\frac{d}{2}+1\right)}.$$

In view of (7) and (41) we find

Corollary 10. *For any open domain $\Omega \subset \mathbb{R}^d$ and any $t > 0$*

$$Z(t) \le \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} - \frac{d\sqrt{\pi}}{\Gamma\left(\frac{d}{2}+1\right)^{\frac{1}{d}}} \frac{|\Omega|^{\frac{d-1}{d}}}{4(4\pi t)^{\frac{d-1}{2}}} + \frac{d(d-1)}{\Gamma\left(\frac{d}{2}+1\right)^{\frac{2}{d}}} \frac{|\Omega|^{\frac{d-2}{d}}}{8(4\pi t)^{\frac{d-2}{2}}}.$$

Remark. The bounds from Corollary 9 and Corollary 10 both capture the main asymptotic behaviour of $Z(t)$ as t tends to zero. Moreover, they contain order-sharp remainder terms. Actually, in the regime $t \to 0+$ the bound form Corollary 10 is stronger than (42). On the other hand the bound from Corollary 10 does not show an exponential decay as t tends to infinity.

Moreover, one can use the ideas of [Mel03] and [HH07] to derive unviersal bounds on $Z(t)$. We can employ inequality (5) and the result of Luttinger (7). For the ball $B_R \subset \mathbb{R}^d$ with $|B_R| = |\Omega|$ the second moment $I(B_R)$ can be calculated explicitly. If we insert the result into (5) we find

$$Z(t) \le \frac{|\Omega|}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\tilde{M}_d \frac{t}{|\Omega|^{\frac{2}{d}}}\right), \tag{43}$$

with a constant $\tilde{M}_d = \frac{d+2}{d}\pi\Gamma\left(\frac{d}{2}+1\right)^{-\frac{2}{d}} M_d$. For example, in dimension $d = 2$ we have $M_2 = \frac{1}{32}$, see [KVW08], and we get

$$Z(t) \le \frac{|\Omega|}{4\pi t} \exp\left(-\frac{\pi}{16}\frac{t}{|\Omega|}\right).$$

In general we have $\tilde{M}_d < 1$ and the estimate (43) is not strong enough to imply the conjectured inequality (6).

But one can employ Corollary 9 to prove (6) at least in low dimensions. To analyse the asymptotic behaviour of the bound from Corollary 9 we refer to the inequalities

$$\begin{aligned} &j_{0,1} && > 2.4 && > \tfrac{1}{\sqrt{\pi}} && \text{if } d = 2 \\ &j_{\frac{1}{2},1} && > 3.1 && > \frac{\Gamma\left(\frac{5}{2}\right)^{\frac{1}{3}}}{\sqrt{\pi}} && \text{if } d = 3 \\ &j_{\frac{d}{2}-1,1} && > \frac{d}{2}-1 && > \frac{\Gamma\left(\frac{d}{2}+1\right)^{\frac{1}{d}}}{\sqrt{\pi}} && \text{if } d \ge 4, \end{aligned}$$

see [AS64]. We find

$$\tilde{\lambda} = \frac{\pi j^2_{\frac{d}{2}-1,1}}{\Gamma\left(\frac{2}{d}+1\right)^{\frac{2}{d}} |\Omega|^{\frac{2}{d}}} > \frac{1}{|\Omega|^{\frac{2}{d}}}.$$

In view of (9) we deduce that (42) is stronger than (6) in the limit $t \to \infty$. Moreover one can employ (13) to show that this relation holds true also in the limit $t \to 0+$. Finally one can compare the bounds for finite values of t numerically and find that (42) is stronger than (6) for all $t > 0$ if $d \le 633$ and that in these dimensions conjecture (6) holds true.

On the other hand numerical evaluations show that for dimensions $d > 633$ there exist $t > 0$ so that the bound in (6) is smaller than the bound in (42). Since the conjecture (6) does not show the expected asymptotic properties we confine ourselves to this numerical discussion.

5. Heat kernel estimates in unbounded domains

In this section we use Proposition 5 to prove upper bounds on $Z(t)$ in unbounded domains, in particular in domains with infinite volume. In such domains, not much is known about universal bounds on $Z(t)$, see [Dav85, Dav89] for results valid in a very general setting. As an example for unbounded domains $\Omega \subset \mathbb{R}^2$, B. Simon and

and M. van den Berg introduced "horn-shaped" regions [Sim83, vdB84a]: Assume $f : [0,\infty) \to [0,\infty)$ is a non-increasing function with $\lim_{s\to\infty} f(s) = 0$ and put

$$\Omega_f = \left\{(x,y) \in \mathbb{R}^2 : x > 0\,,\, 0 < y < f(x)\right\}\,. \tag{44}$$

Then Ω_f is "horn-shaped". Lets state some examples where the short time asymptotics of $Z_f(t)$ can be computed explicitly. Assume $f_\mu(s) = s^{-\frac{1}{\mu}}$, $\mu \geq 1$. Then for $t \to 0+$ we get

$$\begin{aligned} Z(t;\Omega_{f_\mu}) &= \frac{\Gamma\left(1+\frac{\mu}{2}\right)\zeta(\mu)}{2\pi^{\mu+\frac12}}\, t^{-\frac{\mu+1}{2}} + o\left(t^{-\frac{\mu+1}{2}}\right) && \text{if } \mu > 1\,, \\ Z(t;\Omega_{f_1}) &= -\frac{\ln t}{4\pi t} + \frac{1+\gamma-2\ln(2\pi)}{4\pi t} + O\left(t^{-\frac12}\right) && \text{if } \mu = 1\,, \end{aligned} \tag{45}$$

where $\zeta(\mu)$ is the Zeta function and γ denotes Euler's constant, see [Sim83] and [ST90] for refined results. Moreover one can choose $f_e(s) = \exp(-2s)$ and find

$$Z(t;\Omega_{f_e}) = \frac{1}{4\pi t} + \frac{\ln t}{4\sqrt{\pi t}} + O\left(t^{-\frac12}\right) \tag{46}$$

as $t \to 0+$, see [vdB87] and [ST90].

In order to derive universal bounds on $Z(t)$ in unbounded domains, let us first note that all results mentioned in the previous sections, in particular Theorem 1 and Corollary 9 remain valid for unbounded domains $\Omega \subset \mathbb{R}^d$ as long as $|\Omega|$ is finite. Even if the volume of Ω is infinite the estimate (39) holds true as long as Ω_Λ is finite. Moreover, one can use Proposition 5 to estimate $R_\sigma(\Lambda)$ and $Z(t)$ as long as

$$\int_{\frac{\pi}{\sqrt{\Lambda}}}^{\infty} m_i(\tau;\Omega)\, d\tau < \infty \tag{47}$$

for all $\Lambda > 0$. This condition is satisfied for $i = d$ and a suitable choice of coordinate system $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ whenever

$$m_d(\tau;\Omega) = o\left(\tau^{-1}\right)\,, \quad \tau \to \infty\,.$$

For example we can apply Proposition 5 to horn-shaped regions introduced in (44) with $f_\mu(s) = s^{-\frac{1}{\mu}}$, $\mu > 0$.

Theorem 11. *For $\mu > 0$ and all $t > 0$*

$$\begin{aligned} Z(t;\Omega_{f_\mu}) \leq{} & \frac{4}{105\pi^{\frac32}}\frac{1}{\mu-1}\left(\frac{2}{\pi^2}\right)^{\frac{\mu-1}{2}} t^{-\frac{\mu+1}{2}}\,\Gamma\left(\frac{\mu}{2}+4, \frac{\pi^2}{2}t\right) + \frac{1}{4\pi t}\hat\Gamma\left(\frac92, \frac{\pi^2}{2}t\right) \\ & + \frac{4}{105\pi^{\frac32}}\frac{\mu}{1-\mu}\left(\frac{2}{\pi^2}\right)^{\frac{1-\mu}{2\mu}} t^{-\frac{1+\mu}{2\mu}}\,\Gamma\left(\frac{1}{2\mu}+4, \frac{\pi^2}{2}t\right) \end{aligned}$$

if $\mu \neq 1$ and

$$\begin{aligned} Z(t;\Omega_{f_1}) \quad\leq\quad & -\frac{\ln t}{4\pi t}\hat\Gamma\left(\frac92, \frac{\pi^2}{2}t\right) - \frac{1}{4\pi t}\left(2\ln\pi - \ln 2\right)\hat\Gamma\left(\frac92, \frac{\pi^2}{2}t\right) \\ & + \frac{4}{105\pi^{\frac32}t}\int_{\frac{\pi^2}{2}t}^{\infty} s^{\frac72}e^{-s}\ln s\, ds\,. \end{aligned}$$

Proof. In order to apply Proposition 5 choose a coordinate system $(x_1, x_2) \in \mathbb{R}^2$ rotated by $\frac{\pi}{4}$ with respect to the coordinate system (x,y) used in definition (44). Then for $x_1 = 0$ we have

$$p_2\left(0;\Omega_{f_\mu}\right) = \left|\left\{x_2 : (0,x_2) \in \Omega_{f_\mu}\right\}\right|_1 = \sqrt{2}$$

and we find that $m_d\left(\tau;\Omega_{f_\mu}\right)=0$ for all $\tau\geq\sqrt{2}$. Moreover, we can estimate

$$p_2\left(x_1;\Omega_{f_\mu}\right) \;\leq\; \sqrt{2}\;f_\mu\left(\sqrt{2}\,x_1\right) \qquad\qquad \text{if } x_1>0 \qquad \text{and}$$
$$p_2\left(x_1;\Omega_{f_\mu}\right) \;\leq\; \sqrt{2}\;f_\mu^{-1}\left(\sqrt{2}\,|x_1|\right) \qquad\qquad \text{if } x_1<0\,,$$

hence

$$m_d\left(\tau;\Omega_{f_\mu}\right) \;\leq\; 2^{\frac{\mu-1}{2}}\tau^{-\mu}+2^{\frac{1-\mu}{2\mu}}\tau^{-\frac{1}{\mu}}$$

for all $0<\tau<\sqrt{2}$. Inserting these estimates into the inequality from Proposition 5 with $\sigma=\sigma_2=5/2$ yields

$$R_{\frac{5}{2}}(\Lambda) \;=\; 0 \quad \text{for all } 0<\Lambda\leq\frac{\pi^2}{2}$$

and for $\Lambda>\frac{\pi^2}{2}$ we get

$$R_{\frac{5}{2}}(\Lambda) \;\leq\; \frac{1}{14\pi}\left(1-\frac{1}{1-\mu}\,\pi^{1-\mu}\,(2\Lambda)^{\frac{\mu-1}{2}}-\frac{\mu}{\mu-1}\,\pi^{1-\frac{1}{\mu}}\,(2\Lambda)^{\frac{1-\mu}{2\mu}}\right)\Lambda^{\frac{7}{2}}$$

if $\mu\neq1$ and

$$R_{\frac{5}{2}}(\Lambda) \;\leq\; \frac{1}{14\pi}\,(\ln\Lambda+\ln 2-2\ln\pi)\,\Lambda^{\frac{7}{2}}$$

if $\mu=1$. Finally by applying the Laplace transformation to these inequalities and simplifying the resulting estimates on $Z(t)$ we arrive at the claimed results. $\square$

Remark. Comparing these bounds with the asymptotic result (45) we see that the main terms capture the correct order in t as $t\to0+$. In the case $\mu=1$ the first term contains the sharp constant and even the second term is of correct order.

To generalise these considerations to higher dimensions we use slightly different notions. Assume a non-negative function $m(\tau)$ is given for $\tau>0$, right-continuous, non-increasing and satisfying $m(\tau)=o(\tau^{-1})$ as $t\to\infty$. Choose

$$f(s)=\inf\left\{\tau>0: m(\tau)\leq\omega_{d-1}s^{d-1}\right\}\,,$$

where $\omega_{d-1}=\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d+1}{2}\right)^{-1}$ denotes the volume of the unit ball in $\mathbb{R}^{d-1}$, and put

$$\tilde{\Omega}_f=\left\{(x',x_d)\in\mathbb{R}^{d-1}\times\mathbb{R}: |x_d|<\frac{1}{2}f\left(|x'|\right)\right\}\,.$$

Then $\tilde{\Omega}_f$ represents an example of a domain with the distribution function

$$m_d(\tau;\tilde{\Omega}_f) \;=\; m(\tau)\,.$$

In this case, to study explicit examples we choose $f_\mu(s)=\left(\omega_{d-1}s^{d-1}\right)^{-\frac{1}{\mu}}$.

Theorem 12. *For any $\mu>1$ and all $t>0$*

$$Z(t;\tilde{\Omega}_{f_\mu})\leq\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\pi^{1-\mu}}{\mu-1}\frac{\Gamma\left(\sigma_d+\frac{d+\mu+1}{2}\right)}{\Gamma\left(\sigma_d+\frac{d}{2}+1\right)}\;t^{-\frac{d-1+\mu}{2}}\,.$$

Proof. The definition of $\tilde{\Omega}_{f_\mu}$ and the choice of f_μ implies $m_d(\tau,\Omega_{f_\mu})=\tau^{-\mu}$. Hence, we can employ Proposition 5 with $\sigma=\sigma_d$ and find

$$R_{\sigma_d}(\Lambda) \;\leq\; L^{cl}_{\sigma_d,d}\int_{\frac{\pi}{\sqrt{\Lambda}}}^{\infty}\tau^{-\mu}\,d\tau\,\Lambda^{\sigma_d+\frac{d}{2}} \;=\; L^{cl}_{\sigma_d,d}\frac{\pi^{1-\mu}}{\mu-1}\Lambda^{\sigma_d+\frac{d+\mu-1}{2}}\,.$$

To this inequality we can apply the Laplace transformation and simplify the resulting bound on $Z(t)$ to arrive at the claimed result. $\square$

Remark. In dimension $d = 2$, according to [Sim83, vdB87, ST90], the asymptotics (45) and (46) are valid for $\tilde{\Omega}_f$ as well. In this case the bound from Theorem 12 reads as

$$Z(t; \tilde{\Omega}_{f_\mu}) \le \frac{4}{105\,\pi^{\mu+\frac{1}{2}}} \frac{\Gamma\left(4+\frac{\mu}{2}\right)}{\mu-1}\, t^{-\frac{\mu+1}{2}}\,.$$

In view of (45) this bound shows again the correct order in t as $t \to 0+$. Moreover, if we compare the constants

$$b_1(\mu) := \frac{4}{105\,\pi^{\mu+\frac{1}{2}}} \frac{\Gamma\left(4+\frac{\mu}{2}\right)}{\mu-1} \quad \text{and} \quad b_1'(\mu) := \frac{\Gamma\left(1+\frac{\mu}{2}\right)\zeta(\mu)}{2\,\pi^{\mu+\frac{1}{2}}}$$

from the bound above and the asymptotics (45) we find

$$\lim_{\mu\to 1+} \left(\frac{b_1(\mu)}{b_1'(\mu)}\right) = \lim_{\mu\to 1+} \left(\frac{(\mu+6)(\mu+4)(\mu+2)}{(\mu-1)\zeta(\mu)}\right) = 1\,.$$

In order to state an example for unbounded domains with finite volume, choose $f_e(s) = \exp\left(-\omega_{d-1} s^{d-1}\right)$. In the same way as above one can show

Theorem 13. *For all $t > 0$ the estimate*

$$\begin{aligned} Z(t, \tilde{\Omega}_{f_e}) \le{}& \frac{1}{(4\pi t)^{\frac{d}{2}}} \hat{\Gamma}\left(\sigma_d + \frac{d}{2} + 1, \pi^2 t\right) + \frac{\sqrt{\pi}\,\ln t}{4(4\pi t)^{\frac{d-1}{2}}} \frac{\Gamma\left(\sigma_d + \frac{d+1}{2}, \pi^2 t\right)}{\Gamma\left(\sigma_d + \frac{d}{2} + 1\right)} \\ &+ \frac{\sqrt{\pi}\,(\ln\pi - 1)}{2(4\pi t)^{\frac{d-1}{2}}} \frac{\Gamma\left(\sigma_d + \frac{d+1}{2}, \pi^2 t\right)}{\Gamma\left(\sigma_d + \frac{d}{2} + 1\right)} \\ &+ \frac{\sqrt{\pi}}{4(4\pi t)^{\frac{d-1}{2}}} \frac{1}{\Gamma\left(\sigma_d + \frac{d}{2} + 1\right)} \int_{\pi^2 t}^{\infty} s^{\sigma_d + \frac{d}{2}} \ln s\, e^{-s}\, ds \end{aligned}$$

holds true.

Remark. In view of (2) the first term of this bound is sharp in the limit $t \to 0+$ since $|\Omega_{f_e}| = 1$. In dimension $d = 2$ we can use (8) and (46) to point out that even the second term of the bound captures the right order in t as t tends to zero.

6. Proof of Theorem 2

Here we use the results from section 3 to derive universal bounds with correction terms on the Riesz means $R_\sigma(\Lambda)$. First we note that Proposition 5 and Lemma 7 immediately imply the following estimate. Recall that $\tau_\Omega = d^2\pi^2|\Omega|^{-\frac{2}{d}}$ and let $\sigma \ge \frac{3}{2}$ satisfy $\sigma + \frac{d-1}{2} \ge 3$, hence $\delta_{\sigma,d} = 0$. Then for any open domain $\Omega \subset \mathbb{R}^d$ and all $\Lambda > 0$ we find

$$\begin{aligned} R_\sigma(\Lambda) &\le L^{cl}_{\sigma,d} \frac{d-1}{d} |\Omega|\, \Lambda^{\sigma+\frac{d}{2}} && \text{if } \Lambda < \tau_\Omega \quad \text{and} \\ R_\sigma(\Lambda) &\le L^{cl}_{\sigma,d} |\Omega|\, \Lambda^{\sigma+\frac{d}{2}} - \pi\, L^{cl}_{\sigma,d} |\Omega|^{\frac{d-1}{d}} \Lambda^{\sigma+\frac{d-1}{2}} && \text{if } \Lambda \ge \tau_\Omega. \end{aligned}$$

Next we discuss, how a trick by Aizenmann and Lieb [AL78] can be applied to inequalities for eigenvalue means $R_\gamma(\Lambda)$ with remainder terms.

Lemma 14. *Let $\gamma > \sigma \geq \frac{3}{2}$, $\lambda_1 \geq \lambda \geq 0$ and $\Lambda \geq \lambda$. Then*

$$\begin{aligned} R_\gamma(\Lambda) \quad \leq \quad & L^{cl}_{\gamma,d}\, |\Omega|\, \Lambda^{\gamma+\frac{d}{2}} \hat{B}\left(\frac{\lambda}{\Lambda}, \sigma + \frac{d}{2} + 1, \gamma - \sigma\right) \\ & - \frac{L^{cl}_{\sigma,d}}{B\,(\sigma+1,\gamma-\sigma)} \int_0^{\Lambda-\lambda} \tau^{\gamma-\sigma-1} M\left(\frac{\pi}{\sqrt{\Lambda-\tau}};\Omega\right)(\Lambda-\tau)^{\sigma+\frac{d}{2}}\, d\tau \\ & + \frac{\delta_{\sigma,d}}{B\,(\sigma+1,\gamma-\sigma)} \int_0^{\Lambda-\lambda} \tau^{\gamma-\sigma-1} m\left(\frac{\pi}{\sqrt{\Lambda-\tau}};\Omega\right)(\Lambda-\tau)^{\sigma+\frac{d-1}{2}}\, d\tau. \end{aligned} \tag{48}$$

Proof. We start from the well-known identity [AL78]

$$\begin{aligned} R_\gamma(\Lambda) \quad &= \quad \frac{1}{B\,(\sigma+1,\gamma-\sigma)} \int_0^\infty \tau^{\gamma-\sigma-1}\, R_\sigma(\Lambda-\tau)\, d\tau \\ &= \quad \frac{1}{B\,(\sigma+1,\gamma-\sigma)} \int_0^{\Lambda-\lambda} \tau^{\gamma-\sigma-1}\, R_\sigma(\Lambda-\tau)\, d\tau. \end{aligned} \tag{49}$$

Here we have taken into account that $R_\sigma(\tilde{\Lambda}) = 0$ for $\tilde{\Lambda} \leq \lambda \leq \lambda_1$. Now we can apply Proposition 5 and find

$$\begin{aligned} R_\gamma(\Lambda) \quad \leq \quad & \frac{L^{cl}_{\sigma,d}\,|\Omega|}{B\,(\sigma+1,\gamma-\sigma)} \int_0^{\Lambda-\lambda} \tau^{\gamma-\sigma-1}\,(\Lambda-\tau)^{\sigma+\frac{d}{2}}\, d\tau \\ & - \frac{L^{cl}_{\sigma,d}}{B\,(\sigma+1,\gamma-\sigma)} \int_0^{\Lambda-\lambda} \tau^{\gamma-\sigma-1} M\left(\frac{\pi}{\sqrt{\Lambda-\tau}};\Omega\right)(\Lambda-\tau)^{\sigma+\frac{d}{2}}\, d\tau \\ & + \frac{\delta_{\sigma,d}}{B\,(\sigma+1,\gamma-\sigma)} \int_0^{\Lambda-\lambda} \tau^{\gamma-\sigma-1} m\left(\frac{\pi}{\sqrt{\Lambda-\tau}};\Omega\right)(\Lambda-\tau)^{\sigma+\frac{d-1}{2}}\, d\tau. \end{aligned}$$

Finally let us evaluate the first term on the right hand side of this expression. A substitution of the integration variable $s = \frac{\tau}{\Lambda}$ gives

$$\begin{aligned} & \frac{L^{cl}_{\sigma,d}|\Omega|}{B\,(\sigma+1,\gamma-\sigma)}\,\Lambda^{\gamma+\frac{d}{2}} \int_0^{1-\frac{\lambda}{\Lambda}} s^{\gamma-\sigma-1}\,(1-s)^{\sigma+\frac{d}{2}}\, ds \\ = \quad & |\Omega|\,\Lambda^{\gamma+\frac{d}{2}} L^{cl}_{\sigma,d} \frac{B(\gamma-\sigma,\sigma+\frac{d}{2}+1)}{B(\sigma+1,\gamma-\sigma)} \left(1 - \frac{\int_0^{\frac{\lambda}{\Lambda}} (1-t)^{\gamma-\sigma-1} t^{\sigma+\frac{d}{2}} dt}{B(\gamma-\sigma,\sigma+\frac{d}{2}+1)}\right) \\ = \quad & |\Omega|\,\Lambda^{\gamma+\frac{d}{2}} L^{cl}_{\gamma,d} \left(1 - \tilde{B}\left(\tfrac{\lambda}{\Lambda}, \sigma+\tfrac{d}{2}+1, \gamma-\sigma\right)\right). \end{aligned}$$

□

If we apply this Lemma with $\sigma = \sigma_d$ then because of $\delta_{\sigma_d,d} = 0$ the last term on the right hand side of (48) vanishes. This enables us to finish the proof of Theorem 2.

Proof of Theorem 2. Inequality (48) with $\gamma > \sigma = \sigma_d$ and with a substitution $y = \frac{\pi}{\sqrt{\Lambda-\tau}}$ gives

$$\begin{aligned} R_\gamma(\Lambda) \quad \leq \quad & L^{cl}_{\gamma,d}\, |\Omega|\, \Lambda^{\gamma+\frac{d}{2}} \hat{B}\left(\frac{\lambda}{\Lambda}, \sigma_d + \frac{d}{2} + 1, \gamma - \sigma_d\right) \\ & - \frac{2\pi^{2\sigma_d+d+2} L^{cl}_{\sigma_d,d}}{B\,(\sigma_d+1,\gamma-\sigma_d)} \int_{\frac{\pi}{\sqrt{\Lambda}}}^{\frac{\pi}{\sqrt{\lambda}}} \left(\Lambda - \frac{\pi^2}{y^2}\right)^{\gamma-\sigma_d-1} M(y;\Omega)\, y^{-2\sigma_d-d-3}\, dy \end{aligned}$$

for all $\Lambda \geq \lambda$. First we assume $\lambda \geq \tau_\Omega$, i.e. $\frac{\pi}{\sqrt{\lambda}} \leq \frac{1}{d}|\Omega|^{1/d}$. Then we have $y\,|\Omega|^{\frac{d-1}{d}} \leq \frac{1}{d}|\Omega|$ for all $\frac{\pi}{\sqrt{\Lambda}} \leq y \leq \frac{\pi}{\sqrt{\lambda}}$ and in view of Lemma 7 we get

$$\begin{aligned} R_\gamma(\Lambda) \quad &\leq \quad L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+\frac{d}{2}}\hat{B}\left(\frac{\lambda}{\Lambda},\sigma_d+\frac{d}{2}+1,\gamma-\sigma_d\right) \\ &\quad - \frac{2\pi^{2\sigma_d+d+2}L^{cl}_{\sigma_d,d}}{B\left(\sigma_d+1,\gamma-\sigma_d\right)}\,|\Omega|^{\frac{d-1}{d}}\int_{\frac{\pi}{\sqrt{\Lambda}}}^{\frac{\pi}{\sqrt{\lambda}}}\left(\Lambda-\frac{\pi^2}{y^2}\right)^{\gamma-\sigma_d-1} y^{-2\sigma_d-d-2}\,dy\,. \end{aligned}$$

If we substitute $s = \frac{\pi^2}{y^2\Lambda}$ and simplify the expression of the remainder term we arrive at

$$R_\gamma(\Lambda) \;\leq\; L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+\frac{d}{2}}\hat{B}\left(\frac{\lambda}{\Lambda},\sigma_d+\frac{d}{2}+1,\gamma-\sigma_d\right) - S(\Lambda,\lambda)$$

with $S(\Lambda,\lambda)$ as stated in (14).

Next we assume $\lambda < \tau_\Omega$ and proceed in two steps. If at the same time $\Lambda < \tau_\Omega$, that means $\frac{\pi}{\sqrt{\Lambda}} > \frac{1}{d}|\Omega|^{1/d}$, we have $y\,|\Omega|^{\frac{d-1}{d}} > \frac{1}{d}|\Omega|$ for all $\frac{\pi}{\sqrt{\Lambda}} < y\frac{\pi}{\sqrt{\lambda}}$ and by similar calculations as above we arrive at

$$\begin{aligned} R_\gamma(\Lambda) \quad &\leq \quad L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+\frac{d}{2}}\hat{B}\left(\frac{\lambda}{\Lambda},\sigma_d+\frac{d}{2}+1,\gamma-\sigma_d\right) \\ &\quad - \frac{2\pi^{2\sigma_d+d+2}L^{cl}_{\sigma_d,d}}{B\left(\sigma_d+1,\gamma-\sigma_d\right)}\,\frac{|\Omega|}{d}\int_{\frac{\pi}{\sqrt{\Lambda}}}^{\frac{\pi}{\sqrt{\lambda}}}\left(\Lambda-\frac{\pi^2}{y^2}\right)^{\gamma-\sigma_d-1} y^{-2\sigma_d-d-3}\,dy \end{aligned}$$

and we obtain the claimed inequality with $S(\Lambda,\lambda)$ given in (15). On the other hand, if $\lambda < \tau_\Omega \leq \Lambda$ we get

$$\begin{aligned} R_\gamma(\Lambda) \quad &\leq \quad L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+\frac{d}{2}}\hat{B}\left(\frac{\lambda}{\Lambda},\sigma_d+\frac{d}{2}+1,\gamma-\sigma_d\right) \\ &\quad - \frac{2\pi^{2\sigma_d+d+2}L^{cl}_{\sigma_d,d}}{B\left(\sigma_d+1,\gamma-\sigma_d\right)}\,|\Omega|^{\frac{d-1}{d}}\int_{\frac{\pi}{\sqrt{\Lambda}}}^{\frac{|\Omega|^{\frac{1}{d}}}{d}}\left(\Lambda-\frac{\pi^2}{y^2}\right)^{\gamma-\sigma_d-1} y^{-2\sigma_d-d-2}\,dy \\ &\quad - \frac{2\pi^{2\sigma_d+d+2}L^{cl}_{\sigma_d,d}}{B\left(\sigma_d+1,\gamma-\sigma_d\right)}\,\frac{|\Omega|}{d}\int_{\frac{|\Omega|^{\frac{1}{d}}}{d}}^{\frac{\pi}{\sqrt{\lambda}}}\left(\Lambda-\frac{\pi^2}{y^2}\right)^{\gamma-\sigma_d-1} y^{-2\sigma_d-d-3}\,dy. \end{aligned}$$

In this case after a simplification we arrive at $S(\Lambda,\lambda)$ as stated in (16). Finally, if we apply (49) directly to (3) we claim

$$R_\gamma(\Lambda) \;\leq\; L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+\frac{d}{2}}\,\hat{B}\left(\frac{\lambda}{\Lambda},\sigma_d+\frac{d}{2}+1,\gamma-\sigma_d\right)\,.$$

Hence, in the final bound $S(\Lambda,\lambda)$ can be replaced by its positive part $(S(\Lambda,\lambda))_+$. $\square$

Again in view of (11) we can choose

$$\tilde{\lambda} = \frac{\pi j^2_{\frac{d}{2}-1,1}}{\Gamma\left(\frac{d}{2}+1\right)^{2/d}\,|\Omega|^{2/d}}$$

as a lower bound on λ_1. Thus we find

Corollary 15. *Let $\Omega \subset \mathbb{R}^d$ be an open set with finite volume. Then for $\gamma > \sigma_d$ the estimate*

$$R_\gamma(\Lambda) \le L^{cl}_{\gamma,d}\,|\Omega|\,\hat{B}\left(\frac{\tilde{\lambda}}{\Lambda}, \sigma_d + \frac{d}{2} + 1, \gamma - \sigma_d\right)\Lambda^{\gamma+\frac{d}{2}} - (S(\Lambda))_+$$

holds for all $\Lambda \ge \tilde{\lambda}$, where

$$S(\Lambda) = L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+\frac{d}{2}}\frac{1}{d}\,\hat{B}\left(\frac{\tilde{\lambda}}{\Lambda}, \sigma_d + \frac{d}{2} + 1, \gamma - \sigma_d\right)$$

if $\Lambda < \tau_\Omega$ and

$$\begin{aligned} S(\Lambda) = {} & L^{cl}_{\gamma,d-1}\,|\Omega|^{\frac{d-1}{d}}\,\Lambda^{\gamma+\frac{d-1}{2}}\,\frac{B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right)}{2}\,\hat{B}\left(\frac{\tau_\Omega}{\Lambda}, \sigma_d + \frac{d+1}{2}, \gamma - \sigma_d\right) \\ & + L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+\frac{d}{2}}\,\frac{1}{d}\,\tilde{B}\left(\frac{\tilde{\lambda}}{\Lambda}, \frac{\tau_\Omega}{\Lambda}, \sigma_d + \frac{d}{2} + 1, \gamma - \sigma_d\right) \end{aligned}$$

if $\Lambda \ge \tau_\Omega$.

Remark. We can now compare this result with estimate (18) from Proposition 3. In both bounds the high energy asymptotics $\Lambda \to \infty$ is dominated by the sharp first term. In view of (10) also the remainder terms show the correct order as Λ tends to infinity. In this limit the bound from Corollary 15 is stronger than (18) whenever

$$\varepsilon\left(\gamma + \frac{d-1}{2}\right) d_\Lambda(\Omega) < \frac{1}{2}\,B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right)|\Omega|^{\frac{d-1}{d}}$$

holds true. We remark that the right hand side is independent of γ while $\varepsilon\left(\gamma + \frac{d-1}{2}\right)$ tends to zero as γ tends to infinity and $d_\Lambda(\Omega)$ is bounded from above by the diameter of Ω. Hence the condition above will be satisfied for large enough γ.

Moreover, the bound from Corollary 15 contains the factor

$$\hat{B}\left(\frac{\tilde{\lambda}}{\Lambda},\, \sigma_d + \frac{d}{2} + 1, \gamma - \sigma_d\right)$$

which decays exponentially if $\Lambda \to \tilde{\lambda}+$ and which improves the bound from Theorem 2 in comparison to (18) for values of Λ close to $\tilde{\lambda}$.

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